Computability of Banach Space Principles

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Abstract

We investigate the computable content of certain theorems which are sometimes called the "principles" of the theory of Banach spaces. Among these the main theorems are the Open Mapping Theorem, the Closed Graph Theorem and the Uniform Boundedness Theorem. We also study closely related theorems, as Banach's Inverse Mapping Theorem, the Theorem on Condensation of Singularities and the Banach-Steinhaus Theorem. From the computational point of view these theorems are interesting, since their classical proofs rely more or less on the Baire Category Theorem and therefore they count as "non-constructive". These theorems have already been studied in Bishop's constructive analysis but the picture that we can draw in computable analysis differs in several points. On the one hand, computable analysis is based on classical logic and thus can apply stronger principles to prove that certain operations are computable. On the other hand, classical logic enables us to prove "semi-constructive" versions of theorems, which can hardly be expressed in constructive analysis. For instance, the computable version of Banach's Inverse Mapping Theorem states that the inverse T^{-1} of a linear computable and bijective operator T from a computable Banach space into a computable Banach space is computable too, whereas the mapping $T \mapsto T^{-1}$ itself is not computable. Thus, there is no general algorithmic procedure to transfer a program of T into a program of T^{-1}, although a program for T^{-1} always exists. In this way we can explore the border between computability and non-computability in the theory of Banach spaces. As applications we briefly discuss the effective solvability of the initial value problem of ordinary linear differential equations and we prove the existence of computable functions with divergent Fourier series. The focus of our investigation is mainly on infinite-dimensional separable Banach spaces, but also the finite-dimensional case, as well as the non-separable case, will be discussed.

Keywords: computable analysis, functional analysis.

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Contents

1 Introduction

In the twenties and early thirties of the twentieth century a Polish group of mathematicians, headed by Stefan Banach, developed central parts of a discipline which is known as *functional analysis* today. The names of several members of this group, like Steinhaus, Mazur, Schauder, Ulam and others are inseparably connected to well-known functional analytic theorems, like the *Banach-Schauder Theorem* (which is also called *Open Mapping Theorem*), *Banach's Inverse Mapping Theorem* and the *Banach-Steinhaus Theorem* [Ban32].

It is not that well-known that only a bit later Banach and Mazur started to work in *computable analysis*, which is the theory of computable real number functions [BM37]. This discipline had been initiated by Turing, who mentioned as one of his motivations for the invention of his famous machine model the goal to describe real number computations [Tur36]. However, Turing himself did not systematically develop a theory of computable real number functions as it was done later by Banach and Mazur. Unfortunately, the Polish school was scattered by the disorders of the second world war and it was not before the late forties when Mazur gave lectures on computable analysis based on the original work of Banach and himself. Later on Grzegorczyk and Rasiowa compiled a book on this lectures [Maz63]. Because of his own contributions it is Grzegorczyk and the French mathematician Lacombe who count as the main founders of computable analysis, as it is understood nowadays [Grz55, Grz57, Lac55a, Lac55b, Lac55c]. It is a historical curiosity that Banach and Mazur's original definition of a computable function, while suitable for linear operators in Banach spaces (cf. Theorem 8.7), turned out to be to special for the general case of (non-linear) real number functions (cf. [PER89, Her01] for a discussion of different definitions). Today, computable analysis, understood as the theory of computability of real number functions based on classical logic, is represented by several streams, mainly based on the work of Pour-El and Richards [PER89], Ko [Ko91] and Weihrauch [Wei00] and many others. Some aspects of functional analysis, such as the theory of eigenvalues, L_p and Fréchet spaces, distributions and other topics, have already been investigated in this line of research [PER89, Was95, WY96, Was99, YMT99, Zho99, ZZ99, ZW00].

The purpose of this paper is a fusion of two branches of research of Banach's school: basic principles of functional analysis and computability aspects. In other words: we want to study computability aspects of Banach space principles. And we will do this using the representation based approach to computable analysis as it has been developed by Kreitz and Weihrauch and others [Wei00]. This so-called "type-2 theory of effectivity" offers a uniform language to study computability properties of points, sequences, subsets, functions and multi-valued operations and thus, is rich enough to express typical functional analytic theorems. In the following section we will give a brief introduction into the representation based approach to computable analysis and we will introduce those concepts which will be used in later sections. In Section 3 we additionally discuss computable metric spaces and computable Banach spaces and basic examples of such spaces. These definitions are the essential definitions for our investigation since nearly all computable versions of Banach space principles will be expressed using these types of spaces. Since the Open Mapping Theorem and the Closed Graph Theorems require computations with subsets, we have to introduce some hyperspaces and their representations in Section 4. In Sections 5 to 13 we investigate the computational content of several theorems and we study some applications. Especially, we will treat the following theorems:

Principles of the Theory of Banach Spaces:

- the **Open Mapping Theorem**, which states that each linear bounded and surjective operator $T : X \to Y$ is open,

- **Banach's Inverse Mapping Theorem**, which states that each linear bounded and bijective operator $T : X \to Y$ has a bounded inverse T^{-1},

- the **Closed Graph Theorem**, which states that each linear operator $T : X \to Y$ with a closed graph is bounded,

- the **Uniform Boundedness Theorem**, which states that each sequence $(T_n)_{n\in\mathbb{N}}$ of pointwise bounded linear operators $T_n : X \to Y$ is uniformly bounded, and

- the **Banach-Steinhaus Theorem**, which states that each pointwise convergent sequence $(T_n)_{n\in\mathbb{N}}$ of linear operators $T_n : X \to Y$ has a linear bounded operator as pointwise limit.

Additionally, we discuss the Theorem on Condensation of Singularities in Section 10. Later on we will formulate these theorems more precisely; they can also be found in most of the classical textbooks on functional analysis as e.g. [Ban32, DS59, GP65]. The Uniform Boundedness Theorem has first been published in Banach's thesis [Ban22], the Banach-Steinhaus Theorem and the Theorem on Condensation of Singularities in [BS27] (which also includes the Uniform Boundedness Theorem, proved by a Baire Category argument). Banach's Inverse Mapping Theorem has first been published in [Ban29] and the Open Mapping Theorem is due to Schauder [Sch30] (who also gave the first proof of the Inverse Mapping Theorem based on the Open Mapping Theorem). For further historical remarks see [DS59, Heu86].

As an application we show in Section 7 that the computable version of Banach's Inverse Mapping Theorem provides a simple proof of the fact that the solution operator of the initial value problem of a system of ordinary linear differential equations is computable. As a second application we prove in Section 11 that the computable version of the Theorem on Condensation of Singularities implies the existence of a computable function whose Fourier series diverges. In Section 14 we discuss improvements of some results for the finite-dimensional case and in Section 15 we briefly present some results on non-separable spaces. In the conclusions in Section 16 we will briefly compare our results with results from Bishop's school of constructive analysis [BB85] and with Simpson's results on reverse mathematics [Sim99].

Two appendices are devoted to special topics. In Appendix A it will be shown that some of our results can be expressed as results on representations of the space of bounded linear operators. We extend our results in order to obtain a precise picture of the lattice of these representations. Finally, Appendix B discusses computational versions of Baire's Category Theorem. The results of this appendix will be published in [Bra01] and have been included for completeness since they will be applied in Sections 9 and 10.

2 Preliminaries from Computable Analysis

In this section we briefly summarize some notions from computable analysis. For details the interested reader is refered to [Wei00]. The basic idea of the representation based approach to computable analysis is to represent infinite objects like real numbers, functions or sets, by infinite strings over some alphabet Σ (which should at least contain the symbols 0 and 1). Thus, a *representation* of a set X is a surjective mapping $\delta :\subseteq \Sigma^\omega \to X$ and in this situation we will call (X, δ) a *represented space*. Here Σ^ω denotes the set of infinite sequences over Σ and the inclusion symbol is used to indicate that the mapping might be partial. If we have two represented spaces, then we can define the notion of a computable function.

Definition 2.1 (Computable function) Let (X, δ) and (Y, δ') be represented spaces. A function $f :\subseteq X \to Y$ is called (δ, δ')*–computable*, if there exit some computable function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that

$$\delta' F(p) = f\delta(p) \tag{1}$$

for all $p \in \mathrm{dom}(f\delta)$.

The definition is illustrated by the commutative diagram in Figure 1. Of

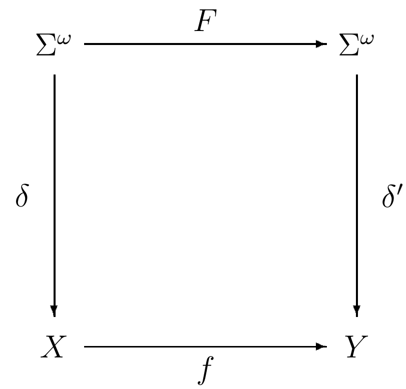

Figure 1: Computability of a function $f : X \to Y$.

course, we have to define computability of functions $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ to make this definition complete, but this can be done via Turing machines: F is computable if there exists some Turing machine, which computes infinitely long and transforms each sequence p, written on the input tape, into the corresponding sequence $F(p)$, written on the one-way output tape. Later on, we will also need computable multi-valued operations $f :\subseteq X \rightrightarrows Y$, which are defined analogously to computable functions by substituting $\delta' F(p) \in f\delta(p)$ for Equation 1 above. If the represented spaces are fixed or clear from the context, then we will simply call a function or operation f *computable*.

For the comparison of representations it will be useful to have the notion of *reducibility* of representations. If δ, δ' are both representations of a set X, then δ is called *reducible* to δ', $\delta \leq \delta'$ in symbols, if there exists a computable function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that $\delta(p) = \delta' F(p)$ for all $p \in \mathrm{dom}(\delta)$. Obviously, $\delta \leq \delta'$ holds, if and only if the identity $\mathrm{id} : X \to X$ is (δ, δ')–computable. Moreover, δ and δ' are called *equivalent*, $\delta \equiv \delta'$ in symbols, if $\delta \leq \delta'$ and $\delta' \leq \delta$.

Analogously to the notion of computability w.r.t. representations we can define the notion of (δ, δ')–*continuity* for single- and multi-valued operations, by substituting a continuous function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ for the computable function F in the definitions above. On Σ^ω we use the *Cantor topology*, which is simply the product topology of the discrete topology on Σ. The corresponding reducibility will be called *continuous reducibility* and we will use the symbols $\leq_t$ and $\equiv_t$ in this case. Again we will simply say that the corresponding function is *continuous*, if the representations are fixed or clear from the context.

This will lead to no confusion with the ordinary topological notion of continuity, as long as we are dealing with *admissible* representations. A representation δ of a topological space X is called *admissible*, if δ is maximal among all continuous representations δ' of X, i.e. if $\delta' \leq_t \delta$ holds for all continuous representations δ' of X. If δ, δ' are admissible representations of T_0–spaces with countable bases, X, Y, then a function $f :\subseteq X \to Y$ is (δ, δ')–continuous, if and only if it is continuous in the ordinary topological sense. For an extension of these notions to larger classes of spaces cf. [Sch00, Sch01].

Given a represented space (X, δ), we will occasionally use the notions of a *computable sequence* and a *computable point*. A *computable sequence* is a computable function $f : \mathbb{N} \to X$, where we assume that $\mathbb{N} = \{0, 1, 2, ...\}$ is represented by $\delta_{\mathbb{N}}(1^n 0^\omega) := n$ and a point $x \in X$ is called *computable*, if there is a constant computable function with value x.

Given two represented spaces (X, δ) and (Y, δ'), there is a canonical representation $[\delta, \delta']$ of $X \times Y$ and a representation $[\delta \to \delta']$ of certain functions $f : X \to Y$. If δ, δ' are *admissible* representations of T_0–spaces with countable bases, then $[\delta \to \delta']$ is actually a representation of the set $\mathcal{C}(X, Y)$ of continuous functions $f : X \to Y$. If $Y = \mathbb{R}$, then we write for short $\mathcal{C}(X) := \mathcal{C}(X, \mathbb{R})$. The function space representation can be characterized by the fact that it admits evaluation and type conversion.

Proposition 2.2 (Evaluation and type conversion) *Let $(X, \delta), (Y, \delta')$ be admissibly represented T_0–spaces with countable bases and let (Z, δ'') be a represented space. Then:*

(1) **(Evaluation)** $\mathrm{ev} : \mathcal{C}(X, Y) \times X \to Y, (f, x) \mapsto f(x)$ *is* $([[\delta \to \delta'], \delta], \delta')$*–computable,*

(2) **(Type conversion)** $f : Z \times X \to Y$*, is* $([\delta'', \delta], \delta')$*–computable, if and only if the function* $\check{f} : Z \to \mathcal{C}(X, Y)$*, defined by* $\check{f}(z)(x) := f(z, x)$ *is* $(\delta'', [\delta \to \delta'])$*–computable.*

The proof of this proposition is based on a version of smn– and utm-Theorem and can be found in [Wei00]. If (X, δ), (Y, δ') are admissibly represented T_0–spaces with countable bases, then in the following we will always assume that $\mathcal{C}(X, Y)$ is represented by $[\delta \to \delta']$. It is known that the computable points in $(\mathcal{C}(X, Y), [\delta \to \delta'])$ are just the (δ, δ')–computable functions $f : X \to Y$ [Wei00]. If (X, δ) is a represented space, then we will always assume that the set of sequences $X^{\mathbb{N}}$ is represented by $\delta^{\mathbb{N}} := [\delta_{\mathbb{N}} \to \delta]$. The computable points in $(X^{\mathbb{N}}, \delta^{\mathbb{N}})$ are just the computable sequences in (X, δ). Moreover, we assume that X^n is always represented by δ^n, which can be defined inductively by $\delta^1 := \delta$ and $\delta^{n+1} := [\delta^n, \delta]$.

Finally, we will call a subset $A \subseteq X$ *δ–r.e.*, if there exists some Turing machine that recognizes A in the following sense: whenever an input $p \in \Sigma^\omega$ with $\delta(p) \in A$ is given to the machine, the machine stops after finitely many steps, for all other $p \in \mathrm{dom}(\delta)$ it computes forever.

3 Computable Metric and Banach Spaces

In this section we will briefly discuss computable metric spaces and computable Banach spaces. The notion of a computable Banach space will be the central notion for all following results. Computable metric spaces have been used in the literature at least since Lacombe [Lac59]. Restricted to computable points they have also been studied by various authors [Ceĭ62, Šan68, Kuš84, Mos64, Spr98]. We consider computable metric spaces as special separable metric spaces but on all points and not only restricted to computable points [Wei93, Wei00]. Pour-El and Richards have introduced a closely related axiomatic characterization of sequential computability structures for Banach spaces [PER89] which has been extended to metric spaces by Mori, Tsujii, and Yasugi and Washihara [MTY97, WY96].

Before we start with the definition of computable metric spaces we just mention that we will denote the *open balls* of a metric space (X,d) by $B(x,\varepsilon) := \{y \in X : d(x,y) < \varepsilon\}$ for all $x \in X$, $\varepsilon > 0$ and correspondingly the *closed balls* by $\overline{B}(x,\varepsilon) := \{y \in X : d(x,y) \leq \varepsilon\}$. Occasionally, we denote complements of sets $A \subseteq X$ by $A^c := X \setminus A$.

Definition 3.1 (Computable metric space) A tuple (X,d,α) is called *computable metric space*, if

(1) $d : X \times X \to \mathbb{R}$ is a metric on X,

(2) $\alpha : \mathbb{N} \to X$ is a sequence which is dense in X,

(3) $d \circ (\alpha \times \alpha) : \mathbb{N}^2 \to \mathbb{R}$ is a computable (double) sequence in $\mathbb{R}$.

Here, we tacitly assume that the reader is familiar with the notion of a computable sequence of reals, but we will come back to that point below. Occasionally, we will say for short that X is a computable metric space. Obviously, a computable metric space is especially separable. Given a computable metric space (X,d,α), its *Cauchy representation* $\delta_X :\subseteq \Sigma^\omega \to X$ can be defined by

$$\delta_X(01^{n_0}01^{n_1}01^{n_2}\ldots) := \lim_{i\to\infty} \alpha(n_i)$$

for all n_i such that $(\alpha(n_i))_{i\in\mathbb{N}}$ converges and $d(\alpha(n_i),\alpha(n_j)) \leq 2^{-i}$ for all $j > i$ (and undefined for all other input sequences). In the following we tacitly assume that computable metric spaces are represented by their Cauchy representation. If X is a computable metric space, then it is easy to see that $d : X \times X \to \mathbb{R}$ becomes computable (see Proposition 3.2 below). All Cauchy representations are admissible w.r.t. the corresponding metric topology.

An important computable metric space is $(\mathbb{R}, d_\mathbb{R}, \alpha_\mathbb{R})$ with the Euclidean metric $d_\mathbb{R}(x,y) := |x-y|$ and some standard numbering of the rational numbers $\mathbb{Q}$, as $\alpha_\mathbb{R}\langle i,j,k\rangle := (i-j)/(k+1)$. Here, $\langle i,j\rangle := 1/2(i+j)(i+j+1)+j$ denotes *Cantor pairs* and this definition is extended inductively to finite tuples. For short we will occasionally write $\overline{k} := \alpha_\mathbb{R}(k)$. In the following we assume that $\mathbb{R}$ is endowed with the Cauchy representation $\delta_\mathbb{R}$ induced by the computable metric space given above.[1] Occasionally, we will also use the represented space $(\mathbb{Q}, \delta_\mathbb{Q})$ of rational numbers with $\delta_\mathbb{Q}(p) := \alpha_\mathbb{R} p(0)$.

Many important representations can be deduced from computable metric spaces, but we will also need some differently defined representations. For instance we will use two further representations $\rho_<, \rho_>$ of the real numbers, which correspond to weaker information on the represented real numbers. Here

$$\rho_<(01^{n_0}01^{n_1}01^{n_2}...) = x :\iff \{q \in \mathbb{Q} : q < x\} = \{\overline{n_i} : i \in \mathbb{N}\}$$

and $\rho_<$ is undefined for all other sequences. Thus, $\rho_<(p) = x$, if p is a list of all rational numbers smaller than x. Analogously, $\rho_>$ is defined with "$>$" instead of "$<$". We write $\mathbb{R}_< = (\mathbb{R}, \rho_<)$ and $\mathbb{R}_> = (\mathbb{R}, \rho_>)$ for the corresponding represented spaces. The computable numbers in $\mathbb{R}_<$ are called *left-computable* real numbers and the computable numbers in $\mathbb{R}_>$ *right-computable* real numbers. The representations $\rho_<$ and $\rho_>$ are admissible w.r.t. to the lower, upper topology on $\mathbb{R}$, which are induced by the open intervals (q, ∞), $(-\infty, q)$, respectively.

Computationally, we do not have to distinguish the complex numbers $\mathbb{C}$ from $\mathbb{R}^2$. Thus, we can directly define a representation of $\mathbb{C}$ by $\delta_\mathbb{C} := \delta_\mathbb{R}^2$. If $z = a + ib \in \mathbb{C}$, then we denote by $\overline{z} := a - ib \in \mathbb{C}$ the *conjugate complex number* and by $|z| := \sqrt{a^2+b^2}$ the *absolute value* of z. Alternatively to the ad hoc definition of $\delta_\mathbb{C}$, we could consider $\delta_\mathbb{C}$ as Cauchy representation of a computable metric space $(\mathbb{C}, d_\mathbb{C}, \alpha_\mathbb{C})$, where $\alpha_\mathbb{C}$ is a numbering of $\mathbb{Q}[i]$, defined by $\alpha_\mathbb{C}\langle n,k\rangle := \overline{n}+\overline{k}i$ and $d_\mathbb{C}(w,z) := |w-z|$ is the Euclidean metric on $\mathbb{C}$. The corresponding Cauchy representation is equivalent to $\delta_\mathbb{R}^2$. In the following we will consider vector spaces over $\mathbb{R}$, as well as over $\mathbb{C}$. We will use the notation $\mathbb{F}$ for a field which always might be replaced by both, $\mathbb{R}$ or $\mathbb{C}$. Correspondingly, we use the notation $(\mathbb{F}, d_\mathbb{F}, \alpha_\mathbb{F})$ for a computable metric space which might be replaced by both computable metric spaces $(\mathbb{R}, d_\mathbb{R}, \alpha_\mathbb{R})$, $(\mathbb{C}, d_\mathbb{C}, \alpha_\mathbb{C})$ defined above. We will also use the notation $Q_\mathbb{F} = \mathrm{range}(\alpha_\mathbb{F})$, i.e. $Q_\mathbb{R} = \mathbb{Q}$ and $Q_\mathbb{C} = \mathbb{Q}[i]$.

The following proposition characterizes the equivalence class of Cauchy representations.

[1] This representation of $\mathbb{R}$ can also be defined, if $(\mathbb{R}, d_\mathbb{R}, \alpha_\mathbb{R})$ just fulfills (1) and (2) of the definition above and this leads to a definition of computable real number sequences without circularity.

Proposition 3.2 (Characterization of the Cauchy representation) *Let (X,d,α) be a recursive metric space with Cauchy representation δ_X and let δ be a further representation of X. Then $\alpha:\mathbb{N}\to X$ is $(\delta_{\mathbb{N}},\delta_X)$–computable and*

(1) $\delta\le\delta_X \iff d:X\times X\to\mathbb{R}$ is $([\delta,\delta_X],\delta_{\mathbb{R}})$–computable,

(2) $\delta_X\le\delta \iff \mathrm{Lim}:\subseteq X^{\mathbb{N}}\to X$ is $(\delta_X^{\mathbb{N}},\delta)$–computable.

Here $\mathrm{Lim}:\subseteq X^{\mathbb{N}}\to X,(x_n)_{n\in\mathbb{N}}\mapsto\lim_{n\to\infty}x_n$ denotes the limit operation of the metric space (X,d), restricted to rapidly converging Cauchy sequences, i.e. $\mathrm{dom}(\mathrm{Lim})=\{(x_n)_{n\in\mathbb{N}}:(\forall i>j)d(x_i,x_j)\le 2^{-j}$ and $(x_n)_{n\in\mathbb{N}}$ converges $\}$. A proof of the previous characterization can be found in [Bra99b]. We proceed with a brief discussion of subspaces and product spaces of computable metric spaces.

Proposition 3.3 (Subspaces) *If (X,d,α) is a computable metric space and $A\subseteq X$ is a subset of X such that there exists a computable sequence $f:\mathbb{N}\to X$ which is dense in A, then the subspace $(A,d|_{A\times A},f)$ is a computable metric space too. The canonical injection $A\hookrightarrow X$ and its inverse are computable.*

Here $d|_{A\times A}$ denotes the metric d restricted to $A\times A$. Especially, the previous proposition implies $\delta_X|_A\equiv\delta_A$ for the corresponding Cauchy representations. The proof is straightforward. The next result shows that computable metric spaces are closed under product in a reasonable way.

Proposition 3.4 (Product spaces) *If (X,d,α), (Y,d',α') are computable metric spaces, then the product space $(X\times Y,d'',\alpha'')$, defined by*

$$d''((x,y),(x',y')):=\max\{d(x,x'),d'(y,y')\}\ \textit{and}\ \alpha''\langle i,j\rangle:=(\alpha(i),\alpha'(j)),$$

is a computable metric space too and the canonical projections of the product space $\mathrm{pr}_1:X\times Y\to X$ and $\mathrm{pr}_2:X\times Y\to Y$ are computable.

Especially, $\delta_{X\times Y}\equiv[\delta_X,\delta_Y]$ holds for the corresponding Cauchy representations. For the definition of a computable Banach space it is helpful to have the notion of a computable vector space which we will define next.

Definition 3.5 (Computable vector space) A represented space (X,δ) is called a *computable vector space* (over $\mathbb{F}$), if $(X,+,\cdot,0)$ is a vector space over $\mathbb{F}$ such that the following conditions hold:

(1) $+:X\times X\to X,(x,y)\mapsto x+y$ is computable,

(2) $\cdot:\mathbb{F}\times X\to X,(a,x)\mapsto a\cdot x$ is computable,

(3) $0 \in X$ is a computable point.

The following proposition lists some examples of computable vector spaces.

Proposition 3.6 (Computable vector spaces) *Let (X, δ) be a computable vector space over $\mathbb{F}$. Then*

(1) $(\mathbb{F}, \delta_{\mathbb{F}})$ *is a computable vector space over $\mathbb{F}$,*

(2) (X^n, δ^n) *is a computable vector space over $\mathbb{F}$,*

(3) $(X^{\mathbb{N}}, \delta^{\mathbb{N}})$ *is a computable vector space over $\mathbb{F}$.*

Let, additionally, (X, δ), (Y, δ') be admissibly represented second countable T_0–spaces. Then

(4) $(\mathcal{C}(Y, X), [\delta' \to \delta])$ *is a computable vector space over $\mathbb{F}$.*

Here we tacitly assume that the vector space operations in product, sequence and function spaces are defined componentwise. The proof for the function space is a straightforward application of evaluation and type conversion. The central definition for the present investigation will be the notion of a computable normed space.

Definition 3.7 (Computable normed space) A tuple $(X, || \; ||, e)$ is called *computable normed space*, if

(1) $|| \; || : X \to \mathbb{R}$ is a norm on X,

(2) $e : \mathbb{N} \to X$ is a *fundamental sequence*, i.e. its linear span is dense in X,

(3) (X, d, α_e) with $d(x, y) := ||x - y||$ and $\alpha_e\langle k, \langle n_0, ..., n_k\rangle\rangle := \sum_{i=0}^{k} \alpha_{\mathbb{F}}(n_i)e_i$, is a computable metric space with Cauchy representation δ_X,

(4) (X, δ_X) is a computable vector space over $\mathbb{F}$.

If in the situation of the definition the underlying space $(X, || \; ||)$ is even a Banach space, i.e. if (X, d) is a complete metric space, then $(X, || \; ||, e)$ is called a *computable Banach space*. If the norm and the fundamental sequence are clear from the context or locally irrelevant, we will say for short that X is a computable normed space or a computable Banach space. We will always assume that computable normed spaces are represented by their Cauchy representations, which are admissible w.r.t. the norm topology. If X is a computable normed space, then $|| \; || : X \to \mathbb{R}$ is a computable function. Of course, all computable Banach space are separable. In the following proposition a number of computable Banach spaces are defined.

Proposition 3.8 (Computable Banach spaces) *Let $p \in \mathbb{R}$ be a computable real number with $1 \leq p < \infty$ and let $a < b$ be computable real numbers. The following spaces are computable Banach spaces over $\mathbb{F}$.*

(1) $(\mathbb{F}^n, || \, ||_p, e)$ *and* $(\mathbb{F}^n, || \, ||_\infty, e)$ *with*

- $||(x_1, x_2, ..., x_n)||_p := \sqrt[p]{\sum_{k=1}^{n} |x_k|^p}$ *and*
  $||(x_1, x_2, ..., x_n)||_\infty := \max_{k=1,...,n} |x_k|$,
- $e_i = e(i) = (e_{i1}, e_{i2}, ..., e_{in})$ *with* $e_{ik} := \begin{cases} 1 & \text{if } (\exists j)\ i = jn + k \\ 0 & \text{else} \end{cases}$.

(2) $(\ell_p, || \, ||_p, e)$ *with*

- $\ell_p := \{x \in \mathbb{F}^{\mathbb{N}} : ||x||_p < \infty\}$,
- $||(x_k)_{k \in \mathbb{N}}||_p := \sqrt[p]{\sum_{k=0}^{\infty} |x_k|^p}$,
- $e_i = e(i) = (e_{ik})_{k \in \mathbb{N}}$ *with* $e_{ik} := \begin{cases} 1 & \text{if } i = k \\ 0 & \text{else} \end{cases}$.

(3) $(L_p, || \, ||_p, e)$ *with*

- $L_p := \{[f] : f : [0,1] \to \mathbb{F} \text{ measurable and } ||[f]||_p < \infty\}$,
- $[f] := \{g : [0,1] \to \mathbb{F} : g \text{ measurable and } ||[f - g]||_p = 0\}$,
- $||[f]||_p := \sqrt[p]{\int_0^1 |f(t)|^p dt}$,
- $e_i = e(i) = [t \mapsto t^i]$.

(4) $(\mathcal{C}[a,b], || \, ||, e)$ *with*

- $\mathcal{C}[a,b] := \{f : [a,b] \to \mathbb{F} : f \text{ continuous}\}$,
- $||f|| := \max_{t \in [a,b]} |f(t)|$,
- $e_i(t) = e(i)(t) = t^i$.

(5) $(\mathcal{C}^{(n)}[a,b], || \, ||_{(n)}, e)$ *with*

- $\mathcal{C}^{(n)}[a,b] := \{f : [a,b] \to \mathbb{F} : f \ n\text{–times continuously differentiable}\}$,
- $||f||_{(n)} := \sum_{i=0}^{n} \max_{t \in [a,b]} |f^{(i)}(t)|$,
- $e_i(t) = e(i)(t) = t^i$.

We leave it to the reader to check that these spaces are actually computable Banach spaces. For the case of L_p, confer e.g. [PER89, ZZ99, Wei00]. If not stated differently, then we will assume that $(\mathbb{F}^n, ||\ ||)$ is endowed with the maximum norm $||\ || = ||\ ||_\infty$. It is known that the Cauchy representation $\delta_{\mathcal{C}[a,b]}$ of $\mathcal{C}[a,b] = \mathcal{C}([a,b],\mathbb{R})$ is equivalent to $[\delta_{[a,b]} \to \delta_\mathbb{R}]$, where $\delta_{[a,b]}$ denotes the restriction of $\delta_\mathbb{R}$ ot $[a,b]$ (cf. Lemma 6.1.10 in [Wei00]). In the following we will often utilize the sequence spaces ℓ_p to construct counterexamples. Since we also want to use some non-separable normed spaces (which cannot be computable by definition) we give some ad hoc definitions for representations of these spaces. Especially, we will deal with the space of bounded linear functions and the space of bounded sequences.

Definition 3.9 (Further normed spaces) Let $(X, ||\ ||), (Y, ||\ ||')$ be computable normed spaces.

(1) Let $\ell_\infty := \{x \in \mathbb{F}^\mathbb{N} : ||x||_\infty < \infty\}$ be endowed with the supremum norm $||(x_k)_{k\in\mathbb{N}}||_\infty := \sup_{k\in\mathbb{N}} |x_k|$ and the representation δ_{ℓ_∞}, defined by

$$\delta_{\ell_\infty}\langle p, q\rangle = x :\iff \delta_\mathbb{F}^\mathbb{N}(p) = x \text{ and } \delta_\mathbb{R}(q) = ||x||_\infty.$$

(2) Let $\mathcal{B}(\mathbb{N}, X) := \{x \in X^\mathbb{N} : ||x|| < \infty\}$ be endowed with the supremum norm $||(x_k)_{k\in\mathbb{N}}|| := \sup_{k\in\mathbb{N}} ||x_k||$ and the representation $\delta_{\mathcal{B}(\mathbb{N},X)}$, defined by

$$\delta_{\mathcal{B}(\mathbb{N},X)}\langle p, q\rangle = x :\iff \delta_X^\mathbb{N}(p) = x \text{ and } \delta_\mathbb{R}(q) = ||x||.$$

(3) Let $\mathcal{B}(X, Y) := \{T \in \mathcal{C}(X, Y) : T \text{ linear}\}$ be endowed with the operator norm $||T|| := \sup_{||x||=1} ||Tx||'$ and the representation $\delta_{\mathcal{B}(X,Y)}$, defined by

$$\delta_{\mathcal{B}(X,Y)}\langle p, q\rangle = T :\iff [\delta_X \to \delta_Y](p) = T \text{ and } \delta_\mathbb{R}(q) = ||T||.$$

Since the space $\mathcal{B}(\mathbb{N}, X)$ in (2) always occurs which $\mathbb{N}$ as source space which will not be considered as linear space, it cannot be confused with the space $\mathcal{B}(X, Y)$ in (3). These definitions are put into a more general framework in Section 15. There it is proved that these spaces are computable Banach spaces in a generalized sense. If X is a computable normed space, then we will use $\delta_{X'} := \delta_{\mathcal{B}(X,\mathbb{F})}$ as standard representation for the dual space $X' := \mathcal{B}(X, \mathbb{F})$, endowed with the operator norm.

We close this section with a brief discussion of subspaces and product spaces of computable normed spaces. The results can essentially be derived from the corresponding results on metric spaces, i.e. Propositions 3.3 and 3.4.

Proposition 3.10 (Subspaces) *If $(X, || \, ||, e)$ is a computable normed space and $A \subseteq X$ is a linear subspace of X such that there exists a computable sequence $f : \mathbb{N} \to X$ whose linear span is dense in A, then the subspace $(A, || \, ||_A, f)$ is a computable normed space too and the canonical injection $A \hookrightarrow X$ and its inverse are computable.*

Here $|| \, ||_A$ denotes the norm $|| \, ||$ restricted to A. The proof is straightforward. Finally, we mention that computable normed spaces are closed under product in a reasonable way.

Proposition 3.11 (Product spaces) *If $(X, || \, ||, e)$, $(Y, || \, ||', e')$ are computable normed spaces, then the product space $(X \times Y, || \, ||'', e'')$, defined by*

$$||(x,y)||'' := \max\{||x||, ||y||'\} \text{ and } e''\langle i,j\rangle := (e(i), e'(j)),$$

is a computable normed space too and the canonical projections of the product space $\mathrm{pr}_1 : X \times Y \to X$ and $\mathrm{pr}_2 : X \times Y \to Y$ are computable.

4 Hyperspaces of Open and Closed Subsets

Since one of our goals is to investigate the Open Mapping Theorem and the Closed Graph Theorem, we have to compute with open and closed sets. Therefore we need representations of the hyperspace $\mathcal{O}(X)$ of open subsets and the hyperspace $\mathcal{A}(X)$ of closed subsets of X. Such representations have been studied in the Euclidean case in [BW99, Wei00] and for the metric case in [BP00].

Definition 4.1 (Hyperspace of open subsets) Let (X, d, α) be a computable metric space. We endow the hyperspace $\mathcal{O}(X) := \{U \subseteq X : U \text{ open}\}$ of open subsets with the representation $\delta_{\mathcal{O}(X)}$, defined by

$$\delta_{\mathcal{O}(X)}(01^{\langle n_0,k_0\rangle}01^{\langle n_1,k_1\rangle}01^{\langle n_2,k_2\rangle}...) := \bigcup_{i=0}^{\infty} B(\alpha(n_i), \overline{k_i}).$$

Any representation of the hyperspace of open subsets induces a representation of the hyperspace of closed subsets. However, in the closed case we will use a further representation and a combination of both.

Definition 4.2 (Hyperspace of closed subsets) Let (X, d, α) be a computable metric space. We endow the hyperspace $\mathcal{A}(X) := \{A \subseteq X : A \text{ closed}\}$ of closed subsets with the representation $\delta^{<}_{\mathcal{A}(X)}$, defined by

$$\begin{aligned} & \delta^{<}_{\mathcal{A}(X)}(01^{\langle n_0,k_0\rangle}01^{\langle n_1,k_1\rangle}01^{\langle n_2,k_2\rangle}...) = A \\ :\iff \quad & \{(n,k) : A \cap B(\alpha(n), \overline{k}) \neq \emptyset\} = \{(n_i, k_i) \in \mathbb{N}^2 : i \in \mathbb{N}\}, \end{aligned}$$

and with the representation $\delta^{>}_{\mathcal{A}(X)}$, defined by $\delta^{>}_{\mathcal{A}(X)}(p) := X \setminus \delta_{\mathcal{O}(X)}(p)$.

Whenever we have two representations δ, δ' of some set, we can define the infimum $\delta \sqcap \delta'$ of δ and δ' by $(\delta \sqcap \delta')\langle p, q\rangle = x :\iff \delta(p) = x$ and $\delta'(q) = x$. We use the short notations $\mathcal{A}_< = \mathcal{A}_<(X) = (\mathcal{A}_<(X), \delta^<_{\mathcal{A}(X)})$, $\mathcal{A}_> = \mathcal{A}_>(X) = (\mathcal{A}_>(X), \delta^>_{\mathcal{A}(X)})$ and $\mathcal{A} = \mathcal{A}(X) = (\mathcal{A}(X), \delta^<_{\mathcal{A}(X)} \sqcap \delta^>_{\mathcal{A}(X)})$ for the corresponding represented spaces. For the computable points of these spaces special names are used.

Definition 4.3 (Recursively enumerable and recursive sets) Let X be a computable metric space and let $A \subseteq X$ be a closed subset.

(1) A is called *r.e. closed*, if A is a computable point in $\mathcal{A}_<(X)$,

(2) A is called *co-r.e. closed*, if A is a computable point in $\mathcal{A}_>(X)$,

(3) A is called *recursive closed*, if A is a computable point in $\mathcal{A}(X)$.

Correspondingly, an open set $U \subseteq X$ is called *r.e., co-r.e., recursive open*, if its complement $X \setminus U$ is a co-r.e., r.e., recursive closed set, respectively.

These definitions generalize the classical notions of r.e. and recursive sets, since a set $A \subseteq \mathbb{N}$, considered as a closed subset of $\mathbb{R}$, is r.e., co-r.e., recursive closed, if and only if it has the same property in the classical sense as a subset of $\mathbb{N}$ [BW99]. We close this section with some helpful results on hyperspaces, which follow directly from results in [BP00]. The first result states that we can represent open subsets by preimages of continuous functions. It is an effective version of the statement that open subsets of metric spaces coincide with the functional open subsets.

Proposition 4.4 (Functional open subsets) *Let X be a computable metric space. The map*

$$Z : \mathcal{C}(X, \mathbb{R}) \to \mathcal{O}(X), f \mapsto X \setminus f^{-1}\{0\}$$

is computable and admits a computable right-inverse $\mathcal{O}(X) \rightrightarrows \mathcal{C}(X, \mathbb{R})$.

The next result can be considered as an effective version of the statement that closed subsets of separable metric spaces are separable again. Here and in the following $\overline{A}$ denotes the topological closure of a subset $A \subseteq X$ of some topological space X.

Proposition 4.5 (Separable closed subsets) *Let X be a computable metric space. The mapping*

$$\overline{\mathrm{range}} : X^{\mathbb{N}} \to \mathcal{A}_<(X), (x_n)_{n\in\mathbb{N}} \mapsto \overline{\{x_n : n \in \mathbb{N}\}}$$

is computable and if X is complete, then it admits a computable multi-valued partial right-inverse $\subseteq \mathcal{A}_<(X) \rightrightarrows X^{\mathbb{N}}$, defined for all non-empty closed subsets.

The next result especially shows that closed subsets $A \subseteq X$ of a metric space (X, d) with a computable *distance function*

$$d_A : X \to \mathbb{R}, x \mapsto \inf_{y \in X} d(x, y)$$

are always recursive closed.

Proposition 4.6 (Located closed subsets) *Let X be a computable metric space. The mapping*

$$L :\subseteq \mathcal{C}(X, \mathbb{R}) \to \mathcal{A}(X), d_A \mapsto A$$

with $\mathrm{dom}(L) := \{f \in \mathcal{C}(X, \mathbb{R}) : (\exists A \in \mathcal{A}(X)) f = d_A\}$ *is computable.*

The inverse is computable only under additional conditions such as local compactness. For a more comprehensive discussion of hyperspaces of computable metric spaces, see [BP00]. We mention the fact that for normed space $\overline{B(x, \varepsilon)} = \overline{B}(x, \varepsilon)$ holds, while in the metric case only "$\subseteq$" holds in general.

5 The Open Mapping Theorem

In this section we will study the effective content of the Open Mapping Theorem, which we formulate first. The proof of this theorem and most of the other Banach space principles which we will study can be found in [GP65] or other textbooks on functional analysis.

Theorem 5.1 (Open Mapping Theorem) *Let X, Y be Banach spaces. If $T : X \to Y$ is a linear surjective and bounded operator, then T is open, i.e. $T(U) \subseteq Y$ is open for any open $U \subseteq X$.*

Whenever $T : X \to Y$ is an open operator, we can associate the function

$$\mathcal{O}(T) : \mathcal{O}(X) \to \mathcal{O}(Y), U \mapsto T(U)$$

with it. Now we can ask for three different computable versions of the Open Mapping Theorem. If $T : X \to Y$ is a linear computable and surjective operator, does the following hold true:

(1) $U \subseteq X$ r.e. open $\Longrightarrow T(U) \subseteq Y$ r.e. open?

(2) $\mathcal{O}(T) : \mathcal{O}(X) \to \mathcal{O}(Y), U \mapsto T(U)$ is computable?

(3) $T \mapsto \mathcal{O}(T)$ is computable?

In the following we will see that questions (1) and (2) can be answered in the affirmative, while question (3) has to be answered in the negative. The key tool for the positive results will be Theorem 5.3 on effective openness. As a preparation of this theorem we will prove that, given an open set $U \in \mathcal{O}(X)$ and a point $x \in U$, we can effectively find some neighborhood of x which is included in U. This statement is made precise by the following lemma.

Lemma 5.2 *Let (X, d, α) be a computable metric space. There exists a computable multi-valued operation $R :\subseteq X \times \mathcal{O}(X) \rightrightarrows \mathbb{N}$ such that for any open $U \subseteq X$ and $x \in U$ there exists some $k \in R(x, U)$ and $B(x, \overline{k}) \subseteq U$ holds for all such k.*

Proof. Given a sequence $\langle n_i, k_i \rangle_{i \in \mathbb{N}}$ of natural numbers such that

$$U = \bigcup_{i=0}^{\infty} B(\alpha(n_i), \overline{k_i})$$

and a sequence $(m_i)_{i \in \mathbb{N}}$ such that $d(\alpha(m_i), \alpha(m_j)) \leq 2^{-j}$ for all $i > j$ and $x := \lim_{i \to \infty} \alpha(m_i) \in U$, there exist $i, j \in \mathbb{N}$ such that $d(\alpha(n_i), \alpha(m_j)) + 2^{-j} < \overline{k_i}$ and thus we can effectively find $i, j, k \in \mathbb{N}$ such that $d(\alpha(n_i), \alpha(m_j)) + 2^{-j} + \overline{k} < \overline{k_i}$ and $\overline{k} > 0$. Then $d(x, y) < \overline{k}$ implies

$$\begin{array}{rcl} d(\alpha(n_i), y) & \leq & d(\alpha(n_i), \alpha(m_j)) + d(\alpha(m_j), x) + d(x, y) \\ & < & d(\alpha(n_i), \alpha(m_j)) + 2^{-j} + \overline{k} \\ & < & \overline{k_i} \end{array}$$

for all $y \in X$ and thus $B(x, \overline{k}) \subseteq B(\alpha(n_i), \overline{k_i}) \subseteq U$. $\square$

Our next theorem on effective openness states that $T : X \to Y$ is computable, if and only if $\mathcal{O}(T) : \mathcal{O}(X) \to \mathcal{O}(Y)$ is computable, provided that T is a linear and bounded operator and X, Y are computable normed spaces.

Theorem 5.3 (Effective openness) *Let X, Y be computable normed spaces and let $T : X \to Y$ be a linear and bounded operator. Then the following conditions are equivalent:*

(1) $T : X \to Y$ is open and computable,

(2) $\mathcal{O}(T) : \mathcal{O}(X) \to \mathcal{O}(Y), U \mapsto T(U)$ is well-defined and computable.

Proof. We consider the computable normed spaces $(X, || \ ||, e)$ and $(Y, || \ ||', e')$ with the dense sequences $\alpha := \alpha_e : \mathbb{N} \to X$, $\beta := \alpha_{e'} : \mathbb{N} \to Y$ according to Definition 3.7. Since no confusion is to be expected, we will also write $|| \ ||$

instead of $|| \, ||'$.

"(1)$\Longrightarrow$(2)" If T is open, then $\mathcal{O}(T)$ is well-defined and we have to prove that $\mathcal{O}(T)$ is computable if T is computable. We separate the proof into two parts (a) and (b). In (a) we use the fact that T is linear and open and in (b) we use the fact that T is computable.

(a) We prove that there exists a computable operation $R :\subseteq X \times \mathcal{O}(X) \rightrightarrows \mathbb{N}$ such that for any open $U \subseteq X$ and $x \in U$ there exists some $k \in R(x, U)$ and $B(Tx, \overline{k}) \subseteq T(U)$ and $\overline{k} > 0$ for all such k. Since T is open and linear, there exists some rational $r > 0$ such that $B(0, r) \subseteq T(B(0, 1))$. Given $U \in \mathcal{O}(X)$ and $x \in U$ we can effectively find some $n \in \mathbb{N}$ with $\varepsilon := \overline{n} > 0$ such that $B(x, \varepsilon) \subseteq U$ by Lemma 5.2 and some $k \in \mathbb{N}$ with $\overline{k} = \varepsilon r$. It follows by linearity of T

$$B(Tx, \overline{k}) = \varepsilon \left(B(0, r) + \frac{1}{\varepsilon} Tx \right) \subseteq \varepsilon \left(TB(0, 1) + \frac{1}{\varepsilon} Tx \right) = TB(x, \varepsilon) \subseteq T(U).$$

Thus, there exists a Turing machine M which computes a realization of R.

(b) Let M' be a Turing machine which computes a (δ_X, δ_Y)–realization of T. We will construct a Turing machine M'' which computes a $(\delta_{\mathcal{O}(X)}, \delta_{\mathcal{O}(Y)})$–realization of $\mathcal{O}(T)$. The set

$$W := \{01^{n_0}01^{n_1}...01^{n_l}0 \in \Sigma^* : l \in \mathbb{N} \text{ and } ||\alpha(n_i) - \alpha(n_j)|| < 2^i \text{ for } i < j \leq l\}$$

is an r.e. subset of Σ^* with $\delta_X(W\Sigma^\omega) = X$. Let $p \in \text{dom}(\delta_{\mathcal{O}(X)})$ with $U := \delta_{\mathcal{O}(X)}(p)$.

Now machine M'' on input p searches systematically for some finite word $w = 01^{n_0}01^{n_1}...01^{n_i}0 \in W$ such that machine M with input $\langle w0^\omega, p \rangle$ produces some (encoded) output $m \in \mathbb{N}$ and M' with input $w0^\omega$ some output v, both after reading only w or some finite prefix of it, such that the following holds: if $01^{k_0}01^{k_1}...01^{k_j}0$ is the longest prefix of v which ends with 0, then $2^{-j+2} < \overline{m}$ and $\alpha(n_i) \in U$. Whenever machine M'' finds such a word w, then it writes $01^{\langle k_j, k \rangle}$ with $\overline{k} = 2^{-j+1}$ on the output tape.

If this happens, then $B(Tx, \overline{m}) \subseteq T(U)$ and $||y - Tx|| \leq 2^{-j}$ with $x := \alpha(n_i)$ and $y := \beta(k_j)$. Thus, $||z - y|| < \overline{k}$ implies

$$||z - Tx|| \leq ||z - y|| + ||y - Tx|| < \overline{k} + 2^{-j} < \overline{m}$$

for all $z \in Y$ and thus $B(\beta(k_j), \overline{k}) \subseteq B(Tx, \overline{m}) \subseteq T(U)$. Hence, we obtain $\delta_{\mathcal{O}(Y)}(q) \subseteq T(U)$ for the output q of M'', provided that this output q is infinite.

It remains to prove that M'' on input p actually produces an infinite output q and that $T(U) \subseteq \delta_{\mathcal{O}(Y)}(q)$. Thus, let $y \in T(U)$. Then there is some $x \in U$ with $Tx = y$ and some $r \in \mathrm{dom}(\delta_X)$ with $\delta_X(r) = x$ such that $r = 01^{n_0}01^{n_1}...$ has infinitely many prefices in W and $\alpha(n_i) \in U$ for all i; especially there is one such prefix $w' \in W$ of r such that M on input $\langle w'0^\omega, p\rangle$ stops with output $m \in \mathbb{N}$ while reading w' or some finite prefix of it and there is some prefix $w \in W$ of r which is longer than w' such that M' on input $w0^\omega$ writes some output $01^{k_0}01^{k_1}...01^{k_j}0$ with $2^{-j+2} < \overline{m}$ while reading w or some finite prefix of it. Finally, M'' will find such a word w and write $01^{\langle k_j, k\rangle}$ with $\overline{k} = 2^{-j+1}$ on the output tape. We obtain $y \in B(\beta(k_j), \overline{k})$ since $||\beta(k_j) - y|| \leq 2^{-j} < \overline{k}$. Moreover, M'' will find infinitely many such words w and produce an infinite output q with $T(U) \subseteq \delta_{\mathcal{O}(Y)}$.

"(2)$\Longrightarrow$(1)" Now let $\mathcal{O}(T)$ be well-defined and computable and let M be a Turing machine which computes a realization of $\mathcal{O}(T)$. Then T is open since $\mathcal{O}(T)$ is well-defined. Since T is bounded, there exists a rational bound $s > 0$ such that $||Tx|| \leq s||x||$ for all $x \in X$ and some $j \in \mathbb{N}$ such that $2^j > s$. We construct a Turing machine M' which computes a (δ_X, δ_Y)–realization of T. Given some input $p = 01^{n_0}01^{n_1}01^{n_2}0...$ with $x := \delta_X(p)$, machine M' works in steps $i = 0, 1, 2, ...$ as follows: in step i machine M' starts machine M with input $q = 01^{\langle n_{i+j+2}, k_i\rangle}01^{\langle n_{i+j+2}, k_i\rangle}01^{\langle n_{i+j+2}, k_i\rangle}0...$ where $\overline{k_i} := 2^{-i-j-2}$ and simulates M until it writes the first word $01^{\langle n,k\rangle}0$. Then M' writes 01^n on its output tape and continues with the next step $i+1$.

Since $\delta_{\mathcal{O}(X)}(q) = B(\alpha(n_{i+j+2}), \overline{k_i})$, M produces an output r with

$$\delta_{\mathcal{O}(Y)}(r) = \mathcal{O}(T)(\delta_{\mathcal{O}(X)}(q)) = TB(\alpha(n_{i+j+2}), \overline{k_i}).$$

Thus, for any subword 01^n which is written by M' in step i on its output tape, we obtain $\beta(n) \in TB(\alpha(n_{i+j+2}), \overline{k_i})$. Since $||x - \alpha(n_{i+j+2})|| \leq \overline{k_i}$, it follows

$$||\beta(n) - Tx|| \leq ||\beta(n) - T\alpha(n_{i+j+2})|| + ||T\alpha(n_{i+j+2}) - Tx|| \leq 2s\overline{k_i} < 2^{-i-1}$$

and hence $\delta_Y(t) = Tx$ holds for the infinite output t of M'. □

Now we can directly conclude a computable version of the Open Mapping Theorem as a corollary of the previous theorem and the classical Open Mapping Theorem.

Corollary 5.4 (Computable Open Mapping Theorem) *Let X, Y be computable Banach spaces and let $T : X \to Y$ be a linear computable operator. If T is surjective, then T is open and $\mathcal{O}(T) : \mathcal{O}(X) \to \mathcal{O}(Y)$ is computable. Especially, $T(U) \subseteq Y$ is r.e. open for any r.e. open set $U \subseteq X$.*

This version of the Open Mapping Theorem leaves open the question whether the map $T \mapsto \mathcal{O}(T)$ itself is computable. Actually, a careful look at the proof of direction "(1)$\Longrightarrow$(2)" of Theorem 5.3 shows that the first step (a) is not effective in T. There, the existence of a rational number $r > 0$ is assumed such that $B(0,r) \subseteq TB(0,1)$. Hence, an effective version of this proof only shows that the map $(T,r) \mapsto \mathcal{O}(T)$ is computable. A similar consideration applies to the direction "(2)$\Longrightarrow$(1)", where an upper bound s on the operator norm $||T|| := \sup_{||x||=1} ||Tx||$ has been used. We formulate both uniform directions of the previous theorem a bit more precisely.

Theorem 5.5 *Let X, Y be computable normed spaces.*

(1) The map[2]

$$\Omega : (T,r) \mapsto \mathcal{O}(T)$$

with $\mathrm{dom}(\Omega) := \{(T,r) : T : X \to Y$ *is linear bounded and open and* $B(0,r) \subseteq TB(0,1)\}$ *is* $([[\delta_X \to \delta_Y], \delta_{\mathbb{Q}}], [\delta_{\mathcal{O}(X)} \to \delta_{\mathcal{O}(Y)}])$*–computable.*

(2) The map

$$\Omega' : (\mathcal{O}(T), s) \mapsto T$$

with $\mathrm{dom}(\Omega') := \{(\mathcal{O}(T), s) : T : X \to Y$ *is linear bounded and open and* $||T|| \leq s\}$ *is* $([[\delta_{\mathcal{O}(X)} \to \delta_{\mathcal{O}(Y)}], \delta_{\mathbb{Q}}], [\delta_X \to \delta_Y])$*–computable.*

The proof is essentially the same as that of Theorem 5.3 with an additional application of the evaluation and type conversion technique, applied to $[\delta_X \to \delta_Y]$, $[\delta_{\mathcal{O}(X)} \to \delta_{\mathcal{O}(Y)}]$, respectively (more precisely, one has to apply the same technique which has been used for the proof of the evaluation and type conversion property, that is a certain smn- and utm-Theorem, cf. Theorem 2.3.13 in [Wei00]). Obviously, the previous theorem holds true for Banach spaces with "surjective" instead of "open", which would be a stronger computable version of the Open Mapping Theorem, but we do not formulate this version of the theorem here.

Now the question appears whether one can prove a uniform version of the Open Mapping Theorem, i.e. whether the map $T \mapsto \mathcal{O}(T)$ is computable without the radius r as additional input information. The answer is "no", even restricted to Banach or Hilbert spaces and to bijective mappings T as we will

[2] We do not mention the functionality of Ω, since this is a bit complicated: if δ, δ' are *admissible* representations of T_0–spaces with countable bases X, Y, then $[\delta \to \delta']$ is an admissible representation of the set $\mathcal{C}(X,Y)$ of continuous functions $X \to Y$ (w.r.t. the compact-open topology, if X is even a Hausdorff space). But if X is not locally compact, then it might happen that $\mathcal{O}(X)$ has no countable base. In such cases we can only assume that $[\delta_{\mathcal{O}(X)} \to \delta_{\mathcal{O}(Y)}]$ is a representation of the set of sequentially continuous functions $\mathcal{O}(X) \to \mathcal{O}(Y)$ (cf. [Sch00]).

show in the following. For the proof we introduce a special type of operators which will serve as a standard counterexample during our investigation. Because of this universality we introduce a specific name.

Lemma 5.6 (Diagonal operators) *Let $p \geq 1$ be a computable real number or $p = \infty$ and let $a \in (0, 1]$. Then $T_a : \ell_p \to \ell_p$ is called a* diagonal operator *of a, if there exists a decreasing sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers with $a_0 = 1$ and $a = \inf_{n\in\mathbb{N}} a_n > 0$, such that*

$$T_a(x_k)_{k\in\mathbb{N}} = (a_k x_k)_{k\in\mathbb{N}}.$$

Each diagonal operator $T_a : \ell_p \to \ell_p$ of a shares the following properties:

(1) $T_a : \ell_p \to \ell_p$ is linear, bounded and bijective,

(2) $||T_a|| = 1$ and $||T_a^{-1}|| = \frac{1}{a}$,

(3) $B(0, r) \subseteq T_a B(0, 1)$ implies $r \leq a$ for all $r > 0$.

Proof. (1) Obviously, T_a is linear. In case $1 \leq p < \infty$ we obtain

$$||T_a(x_k)_{k\in\mathbb{N}}||_p = \sqrt[p]{\sum_{k=0}^{\infty} |a_k x_k|^p} \leq a_0 \cdot \sqrt[p]{\sum_{k=0}^{\infty} |x_k|^p} = ||(x_k)_{k\in\mathbb{N}}||_p$$

and in case $p = \infty$ we obtain

$$||T_a(x_k)_{k\in\mathbb{N}}||_\infty = \sup_{k\in\mathbb{N}} |a_k x_k| \leq a_0 \cdot \sup_{k\in\mathbb{N}} |x_k| = ||(x_k)_{k\in\mathbb{N}}||_\infty$$

Thus, $||T_a|| \leq 1$ and T_a is bounded in both cases. Moreover, T_a is obviously injective since $a_n > 0$ for all $n \in \mathbb{N}$ and T_a is surjective since $(y_k)_{k\in\mathbb{N}} \in \ell_p$ implies $||(y_k)_{k\in\mathbb{N}}||_p < \infty$ and thus

$$\left|\left|\left(\frac{1}{a_k} y_k\right)_{k\in\mathbb{N}}\right|\right|_p = \sqrt[p]{\sum_{k=0}^{\infty} \left|\frac{1}{a_k} y_k\right|^p} \leq \frac{1}{a} \cdot ||(y_k)_{k\in\mathbb{N}}||_p < \infty$$

in case $1 \leq p < \infty$ and analogously $||(\frac{1}{a_k} y_k)_{k\in\mathbb{N}}||_\infty \leq \frac{1}{a} \cdot ||(y_k)_{k\in\mathbb{N}}||_\infty < \infty$ and hence $(\frac{1}{a_k} y_k)_{k\in\mathbb{N}} \in \ell_p$ and $T_a(\frac{1}{a_k} y_k)_{k\in\mathbb{N}} = (y_k)_{k\in\mathbb{N}}$ in all cases. Moreover, this already shows $||T_a^{-1}|| \leq \frac{1}{a}$.

(2) Now let $e_{ik} \in \mathbb{F}$ be defined as in Proposition 3.8(2). Then $e_i := (e_{ik})_{k\in\mathbb{N}} \in \ell_p$ is a unit vector for each $i \in \mathbb{N}$ and $T_a(e_i) = a_i e_i$ and $T_a^{-1}(e_i) = \frac{1}{a_i} e_i$ for all

$i \in \mathbb{N}$. This proves $||T_a|| \geq a_i$ and $||T_a^{-1}|| \geq \frac{1}{a_i}$ for all $i \in \mathbb{N}$ and thus altogether $||T_a|| = a_0 = 1$ and $||T_a^{-1}|| = \frac{1}{a}$.

(3) Finally, let $B(0,r) \subseteq T_a B(0,1)$. Now $se_i \in B(0,r)$ for all positive $s < r$ and $i \in \mathbb{N}$ and hence $T_a^{-1}(se_i) = \frac{s}{a_i} e_i \in B(0,1)$ and thus $s < a_i$ for all positive $s < r$ and $i \in \mathbb{N}$. Hence, $r \leq a_i$ for all $i \in \mathbb{N}$ and thus $r \leq a = \inf_{i\in\mathbb{N}} a_i$. $\square$

The next proposition shows that we can effectively find a diagonal operator $T_a : \ell_p \to \ell_p$ for any given $a \in \mathbb{R}_>$ with $0 < a \leq 1$.

Proposition 5.7 (Diagonal operator) *Let $p \geq 1$ be a computable real number. There exists a computable multi-valued operation $\tau :\subseteq \mathbb{R}_> \rightrightarrows \mathcal{B}(\ell_p, \ell_p)$ such that for any $a \in \mathbb{R}_>$ with $a \in (0,1]$ there exists some $T_a \in \tau(a)$ and all such $T_a : \ell_p \to \ell_p$ are diagonal operators of a.*

Proof. Given a real number $a \in \mathbb{R}_>$ with $a \in (0,1]$, we can effectively find a decreasing sequence $(a_n)_{n\in\mathbb{N}}$ of rational numbers $a_n \in \mathbb{Q}$ such that $a_0 = 1$ and $a = \inf_{n\in\mathbb{N}} a_n$. We define a diagonal operator $T_a : \ell_p \to \ell_p$ of a by $T_a(x_k)_{k\in\mathbb{N}} := (a_k x_k)_{k\in\mathbb{N}}$ for all $(x_k)_{k\in\mathbb{N}} \in \ell_p$. Given some $x = (x_k)_{k\in\mathbb{N}}$ and a precision $m \in \mathbb{N}$ we can effectively find some $n \in \mathbb{N}$ and numbers $q_0, ..., q_n \in Q_\mathbb{F}$ such that $||\sum_{i=0}^n q_i e_i - x||_p < 2^{-m}$. Since $||T_a|| = 1$ by Lemma 5.6(2), it follows

$$\left|\left|T_a\left(\sum_{i=0}^n q_i e_i\right) - T_a(x)\right|\right|_p \leq ||T_a|| \cdot \left|\left|\sum_{i=0}^n q_i e_i - x\right|\right|_p < 2^{-m}$$

By linearity of T_a we obtain $T_a(\sum_{i=0}^n q_i e_i) = \sum_{i=0}^n q_i T_a(e_i) = \sum_{i=0}^n q_i a_i$ and thus we can evaluate T_a effectively up to any given precision m. Using type conversion we can actually prove that there exists an operation τ with the desired properties. Here it should be mentioned that $||T_a|| = 1$ holds for all diagonal operators T_a, and thus we can translate $[\delta_{\ell_p} \to \delta_{\ell_p}]$ to $\delta_{\mathcal{B}(\ell_p,\ell_p)}$ for such operators. $\square$

Now we can prove the promised negative result on the mapping $T \mapsto \mathcal{O}(T)$.

Theorem 5.8 *Let $p \geq 1$ be a computable[3] real number. Then the mapping $T \mapsto \mathcal{O}(T)$, defined for linear, bounded and bijective operators $T : \ell_p \to \ell_p$ with $||T|| = 1$, is not continuous.*

[3]Here and in the following, discontinuity results also hold true for non-computable p. But in order to keep the proofs as simple as possible, we only formulate the versions for computable p. Otherwise we had to add pure continuity versions for arbitrary $p \geq 1$ of those positive results, like Proposition 5.7, which are used to derive discontinuity statements.

Proof. Let us assume that the mapping $T \mapsto \mathcal{O}(T)$, defined for linear, bounded and bijective operators $T : \ell_p \to \ell_p$ with $||T|| = 1$ would be continuous, more precisely, $([\delta_{\ell_p} \to \delta_{\ell_p}], [\delta_{\mathcal{O}(\ell_p)} \to \delta_{\mathcal{O}(\ell_p)}])$–continuous. Then by the evaluation property, $E : T \mapsto TB(0,1)$ is $([\delta_{\ell_p} \to \delta_{\ell_p}], \delta_{\mathcal{O}(\ell_p)})$–continuous. Now the operation $S :\subseteq \mathbb{R}_> \rightrightarrows \mathbb{N}$, defined by $S(a) := R(0, E \circ \tau(a))$, where $R :\subseteq \ell_p \times \mathcal{O}(\ell_p) \rightrightarrows \mathbb{N}$ is the computable operation from Lemma 5.2 and $\tau :\subseteq \mathbb{R}_> \rightrightarrows \mathcal{C}(\ell_p, \ell_p)$ is the computable operation from Proposition 5.7, is continuous too. Thus, for any positive $a \in \mathbb{R}_>$, there exists some $n \in S(a)$ and for all such n we obtain $0 < \overline{n} \leq a$ by construction of τ and Lemma 5.6(3). In other words, S determines a positive lower rational bound for any $a \in \mathbb{R}_>$ with $0 < a \leq 1$. It is straightforward to see that such a continuous operation S cannot exist. □

Although the mapping $T \mapsto \mathcal{O}(T)$ is discontinuous, we know by Theorem 5.3 that $\mathcal{O}(T) : \mathcal{O}(\ell_p) \to \mathcal{O}(\ell_p)$ is computable whenever $T : \ell_p \to \ell_p$ is computable. On the one hand, this guarantees that $T \mapsto \mathcal{O}(T)$ is not too discontinuous [Bra99a]. On the other hand, we have to use sequences to construct a computable counterexample for the uniform version of the Open Mapping Theorem. Therefore we will use the following lemma which can be proved by an easy diagonalization argument.

Lemma 5.9 *There exists a computable sequence $(a_n)_{n\in\mathbb{N}}$ of positive right-computable real numbers such that for any computable function $f : \mathbb{N} \to \mathbb{N}$ there exists some $i \in \mathbb{N}$ such that $2^{-f(i)} > a_i$.*

Proof. We use some total Gödel numbering $\varphi : \mathbb{N} \to P$ of the set of partial computable functions $P := \{f :\subseteq \mathbb{N} \to \mathbb{N} : f \text{ computable}\}$ to define

$$a_n := \begin{cases} 2^{-k-1} & \text{if } \varphi_n(n) = k \\ 1 & \text{if } \varphi_n(n) \text{ is undefined} \end{cases}$$

for all $n, k \in \mathbb{N}$. Then $(a_n)_{n\in\mathbb{N}}$ is a computable sequence of positive right-computable real numbers. Now let $f : \mathbb{N} \to \mathbb{N}$ be some total computable function. Then there exists some $i \in \mathbb{N}$ such that $\varphi_i = f$ and we obtain $2^{-f(i)} > 2^{-\varphi_i(i)-1} = a_i$. □

Using the computable sequence $(a_n)_{n\in\mathbb{N}}$ of right-computable real numbers constructed in this lemma and the computable operation $\tau :\subseteq \mathbb{R}_> \rightrightarrows \mathcal{B}(\ell_p, \ell_p)$ from Proposition 5.7, we directly obtain a computable sequence $(T_n)_{n\in\mathbb{N}}$ of operators $T_n \in \tau(a_n)$, i.e. T_n is a diagonal operator of a_n, with the following property.

Corollary 5.10 *Let p be a computable real number with $p \geq 1$. There exists a computable sequence $(T_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(\ell_p, \ell_p)$ of linear computable and bijective operators $T_n : \ell_p \to \ell_p$ such that $(T_n B(0,1))_{n\in\mathbb{N}}$ is a sequence of r.e. open subsets of ℓ_p which is not computable.*

If the sequence $(T_n B(0,1))_{n\in\mathbb{N}}$ would be computable, then the computable operation $R :\subseteq \ell_p \times \mathcal{O}(\ell_p) \rightrightarrows \mathbb{N}$ from Lemma 5.2 would yield a computable sequence $(r_n)_{n\in\mathbb{N}}$ of rationals with $r_n \in R(0, T_n B(0,1))$ such that $0 < r_n \leq a_n$ by Lemma 5.6(3). But this contradicts Lemma 5.9.

6 Banach's Inverse Mapping Theorem

In this section we want to study computable versions of Banach's Inverse Mapping Theorem. Again we start with a formulation of the classical theorem.

Theorem 6.1 (Banach's Inverse Mapping Theorem) *Let X, Y be Banach spaces and let $T : X \to Y$ be a linear bounded operator. If T is bijective, then $T^{-1} : Y \to X$ is bounded.*

It is clear that linearity is an essential property in this theorem since it is well-known that there are continuous bijections with discontinuous inverse. Similarly as in case of the Open Mapping Theorem we have two canonical candidates for an effective version of this theorem. We can ask whether for linear bounded and bijective operators $T : X \to Y$ the following holds true:

(1) T computable $\Longrightarrow T^{-1}$ computable?

(2) $T \mapsto T^{-1}$ is computable?

Again we will see that the first question has to be answered in the affirmative and the second question has to be answered in the negative. Analogously as we have used Theorem 5.3 on effective openness to prove the computable Open Mapping Theorem 5.4, we will use Theorem 6.2 on effective continuity to prove the computable version of Banach's Inverse Mapping Theorem. The first part of the proof is based on Proposition 4.4.

Theorem 6.2 *Let X, Y be computable metric spaces and let $T : X \to Y$ be a function. Then the following is equivalent:*

(1) $T : X \to Y$ is computable,

(2) $\mathcal{O}(T^{-1}) : \mathcal{O}(Y) \to \mathcal{O}(X), V \mapsto T^{-1}(V)$ is well-defined and computable.

Proof. "(1)$\Longrightarrow$(2)" If $T : X \to Y$ is computable, then it is continuous and hence $\mathcal{O}(T^{-1})$ is well-defined. Given a function $f : Y \to \mathbb{R}$ such that $V = Y \setminus f^{-1}\{0\}$, we obtain

$$T^{-1}(V) = T^{-1}(Y \setminus f^{-1}\{0\}) = X \setminus (fT)^{-1}\{0\}.$$

Using Proposition 4.4 and the fact that composition

$$\circ : \mathcal{C}(Y, \mathbb{R}) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, \mathbb{R}), (f, T) \mapsto f \circ T$$

is computable (which can be proved by evaluation and type conversion), we obtain that $\mathcal{O}(T^{-1})$ is computable.

"(2)$\Longrightarrow$(1)" We consider the computable metric spaces (X, d, α) and (Y, d', β). We note that T is continuous, if $\mathcal{O}(T^{-1})$ is well-defined. Given a Turing machine M which computes a realization of $\mathcal{O}(T^{-1})$, we construct a Turing machine M' which computes a realization of T. The machine M' with input $p \in \mathrm{dom}(\delta_X)$ works in steps $k = 0, 1, 2, \ldots$ as follows. In step k machine M' simultaneously tests all values $n \in \mathbb{N}$ until some value is found with the following property: machine M with input $01^{\langle n,m\rangle}01^{\langle n,m\rangle}01^{\langle n,m\rangle}0\ldots$ with $\overline{m} = 2^{-k}$ produces an output with subword $01^{\langle i,j\rangle}0$ such that $x = \delta_X(p) \in B(\alpha(i), \overline{j})$. As soon as such a subword is found, M' writes 01^n on the output tape.

If this happens, i.e. if M' writes 01^n on its output tape, then we obtain $Tx \in B(\beta(n), 2^{-k})$ since $x \in B(\alpha(i), \overline{j}) \subseteq T^{-1}(B(\beta(n), \overline{m}))$. Moreover, M' actually produces an infinite output q, since for any $k \in \mathbb{N}$ there is some $n \in \mathbb{N}$ such that $Tx \in B(\beta(n), 2^{-k})$ and thus $x \in T^{-1}(B(\beta(n), 2^{-k}))$ and consequently M on input $01^{\langle n,m\rangle}01^{\langle n,m\rangle}0\ldots$ has to produce some output with subword $01^{\langle i,j\rangle}0$ and $x \in B(\alpha(i), \overline{j})$. It follows $\delta_Y(q) = Tx$. $\square$

Now we note the fact that for Banach spaces X, Y and bijective linear operators $T : X \to Y$, the operation $\mathcal{O}(T^{-1})$, associated with T according to the previous theorem, is the same as the operation $\mathcal{O}(S)$, associated with $S = T^{-1}$ according to Corollary 5.4. Thus, we can directly conclude the following computable version of the Inverse Mapping Theorem as a corollary of Theorem 6.2 and Corollary 5.4.

Corollary 6.3 (Computable Inverse Mapping Theorem) *Let X, Y be computable Banach spaces and let $T : X \to Y$ be a linear computable operator. If T is bijective, then $T^{-1} : Y \to X$ is computable too.*

Analogously to the Open Mapping Theorem, we want to study the question whether a uniform computable version of Banach's Inverse Mapping Theorem holds true. Unfortunately, the answer is "no" in this case too, i.e. the mapping $T \mapsto T^{-1}$ is discontinuous. Before proving this negative result we formulate a uniform version of Theorem 6.2. This version can be proved correspondingly to Theorem 6.2 simply by using evaluation and type conversion (certain smn- and utm-Theorems, respectively).

Theorem 6.4 (Effective Continuity) *Let X, Y be computable metric spaces. Then the total map*

$$\omega : T \mapsto \mathcal{O}(T^{-1}),$$

defined for all continuous $T : X \to Y$, is $([\delta_X \to \delta_Y], [\delta_{\mathcal{O}(Y)} \to \delta_{\mathcal{O}(X)}])$–computable and its inverse ω^{-1} is computable in the corresponding sense too.

Using this positive result, applied to T^{-1}, we can transfer our negative results on the Open Mapping Theorem to the Inverse Mapping Theorem. As a corollary of the previous Theorem 6.4 and Theorem 5.8 we obtain the following result.

Corollary 6.5 *Let $p \geq 1$ be a computable real number. The inversion map $T \mapsto T^{-1}$, defined for linear bounded and bijective operators $T : \ell_p \to \ell_p$ with $||T|| = 1$, is not continuous.*

This holds true w.r.t. $([\delta_{\ell_p} \to \delta_{\ell_p}], [\delta_{\ell_p} \to \delta_{\ell_p}])$–continuity, as well as w.r.t. $(\delta_{\mathcal{B}(\ell_p,\ell_p)}, [\delta_{\ell_p} \to \delta_{\ell_p}])$– and $(\delta_{\mathcal{B}(\ell_p,\ell_p)}, \delta_{\mathcal{B}(\ell_p,\ell_p)})$–continuity. On the other hand, a combination of this corollary with the statement on the inverse ω^{-1} in Theorem 6.4 yields the following additional negative result which corresponds to Theorem 5.5(2).

Corollary 6.6 *Let $p \geq 1$ be a computable real number. The map $\mathcal{O}(T) \mapsto T$, defined for linear bounded and bijective operators $T : \ell_p \to \ell_p$ with $||T|| = 1$, is not continuous.*

Correspondingly, we can construct a computable counterexample for the uniform version of the Inverse Mapping Theorem. As a corollary of Theorem 6.4, applied to T_n^{-1} from Corollary 5.10 we obtain the following counterexample.

Corollary 6.7 *For any computable real number $p \geq 1$ there exists a computable sequence $(T_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(\ell_p, \ell_p)$ of linear computable and bijective operators $T_n : \ell_p \to \ell_p$, such that $(T_n^{-1})_{n\in\mathbb{N}}$ is a sequence of computable operators $T_n^{-1} : \ell_p \to \ell_p$ which is not computable in $\mathcal{C}(\ell_p, \ell_p)$.*

We can even assume that $||T_n|| = 1$ for all $n \in \mathbb{N}$. After this negative result, which shows that the mapping $T \mapsto T^{-1}$ is not computable, we want to discuss which additional input information on T could help to establish a uniform computable version of Inverse Mapping Theorem. Theorem 5.5 together with Theorem 6.4 constitute two formally different ways to prove such a result, which correspond to the compositions

$$(T, r) \stackrel{\Omega}{\longmapsto} \mathcal{O}(T) \stackrel{\omega^{-1}}{\longmapsto} T^{-1} \quad \text{and} \quad (T, s) \stackrel{\omega \times \mathrm{id}}{\longmapsto} (\mathcal{O}(T^{-1}), s) \stackrel{\Omega'}{\longmapsto} T^{-1}.$$

In the first case, as additional input information a rational radius $r > 0$ is required such that $B(0, r) \subseteq TB(0, 1)$ and in the second case a rational bound $s > 0$ such that $||T^{-1}|| \leq s$. It is easy to see that both types of additional input information are equivalent in this situation. On the one hand, this follows by applying the following lemma to the inverse T^{-1}.

Lemma 6.8 *Let X, Y be normed spaces and let $T : X \to Y$ be a linear bounded map. Then*

$$B(0, r) \subseteq T^{-1}B(0, 1) \implies ||T|| \leq \frac{1}{r}$$

holds for all $r > 0$.

Proof. By linearity and continuity of T we obtain

$$\begin{aligned} B(0, r) \subseteq T^{-1}B(0, 1) &\implies TB(0, 1) \subseteq B\left(0, \frac{1}{r}\right) \\ &\implies T\overline{B}(0, 1) \subseteq \overline{B}\left(0, \frac{1}{r}\right) \\ &\implies ||T|| = \sup_{x \in \overline{B}(0,1)} ||Tx|| \leq \frac{1}{r}. \end{aligned}$$

□

On the other hand, $||T^{-1}|| \leq s$ implies $T^{-1}B(0, 1) \subseteq B\left(0, \frac{1}{r}\right)$, whenever $s < \frac{1}{r}$. For completeness we formulate one of the two equivalent uniform versions of the Inverse Mapping Theorem precisely.

Theorem 6.9 (Inversion) *Let X, Y be computable normed spaces. The map*

$$\iota : (T, s) \mapsto T^{-1}$$

with $\mathrm{dom}(\iota) := \{(T, s) : T : X \to Y$ *is linear, bounded and bijective and* $||T^{-1}|| \leq s\}$ *is* $([[\delta_X \to \delta_Y], \delta_{\mathbb{Q}}], [\delta_Y \to \delta_X])$*–computable.*

We close this section with an application of the Computable Inverse Mapping Theorem 6.3 which shows that any two comparable computable complete norms are computably equivalent.

Theorem 6.10 *Let* $(X, ||\ ||)$, $(X, ||\ ||')$ *be computable Banach spaces and let* δ, δ' *be the corresponding Cauchy representations of* X. *If* $\delta \leq \delta'$ *then* $\delta \equiv \delta'$.

Proof. If $\delta \leq \delta'$, then the identity $\mathrm{id} : (X, ||\ ||) \to (X, ||\ ||')$ is (δ, δ')–computable. Moreover, the identity is obviously linear and bijective. Thus, $\mathrm{id}^{-1} : (X, ||\ ||') \to (X, ||\ ||)$ is (δ', δ)–computable by the Computable Inverse Mapping Theorem. Consequently, $\delta' \leq \delta$. □

7 Initial Value Problem

In this section we will discuss an application of the computable version of Banach's Inverse Mapping Theorem to the initial value problem of ordinary linear differential equations. Consider the linear differential equation with initial values

$$\sum_{i=0}^{n} f_i(t) x^{(i)}(t) = y(t) \text{ with } x^{(j)}(0) = a_j \text{ for } j = 0, ..., n-1.$$

Here, $x, y : [0, 1] \to \mathbb{R}$ are functions, $f_i : [0, 1] \to \mathbb{R}$ are coefficient functions with $f_n \neq 0$ and $a_0, ..., a_{n-1} \in \mathbb{R}$ are initial values. It is known that for each $y \in \mathcal{C}[0, 1]$ and all values $a_0, ..., a_{n-1}$ there is exactly one solution $x \in \mathcal{C}^{(n)}[0, 1]$ of this equation [Heu86]. Given f_i, a_i and y, can we effectively find this solution? The positive answer to this question can easily be deduced from the computable Inverse Mapping Theorem 6.3.

Theorem 7.1 (Initial Value Problem) *Let* $n \geq 1$ *be a natural number and let* $f_0, ..., f_n : [0, 1] \to \mathbb{R}$ *be computable functions with* $f_n \neq 0$. *The solution operator*

$$L : \mathcal{C}[0, 1] \times \mathbb{R}^n \to \mathcal{C}^{(n)}[0, 1]$$

which maps each tuple $(y, a_0, ..., a_{n-1}) \in \mathcal{C}[0, 1] \times \mathbb{R}^n$ *to the unique function* $x = L(y, a_0, ..., a_{n-1})$ *with*

$$\sum_{i=0}^{n} f_i(t) x^{(i)}(t) = y(t) \textit{ with } x^{(j)}(0) = a_j \textit{ for } j = 0, ..., n-1,$$

is computable.

Proof. The operator

$$L^{-1} : \mathcal{C}^{(n)}[0,1] \to \mathcal{C}[0,1] \times \mathbb{R}^n, x \mapsto \left(\sum_{i=0}^{n} f_i x^{(i)}, x^{(0)}(0), ..., x^{(n-1)}(0) \right)$$

is obviously linear. Using the evaluation and type conversion property and the fact that the i–th differentiation operator $\mathcal{C}^{(n)}[0,1] \to \mathcal{C}[0,1], x \mapsto x^{(i)}$ is computable for $i \leq n$, one can easily prove that L^{-1} is computable. By the computable Inverse Mapping Theorem 6.3 it follows that L is computable too. □

We obtain the following immediate corollary on computability of solutions of ordinary linear differential equations.

Corollary 7.2 *Let $n \geq 1$ and let $y, f_0, ..., f_n : [0,1] \to \mathbb{R}$ be computable functions and let $a_0, ..., a_{n-1} \in \mathbb{R}$ be computable real numbers. Then the unique function $x \in \mathcal{C}^{(n)}[0,1]$ with*

$$\sum_{i=0}^{n} f_i(t) x^{(i)}(t) = y(t) \text{ with } x^{(j)}(0) = a_j \text{ for } j = 0, ..., n-1,$$

is a computable point in $\mathcal{C}^{(n)}[0,1]$. Especially, $x^{(0)}, ..., x^{(n)} : [0,1] \to \mathbb{R}$ are computable functions.

8 The Closed Graph Theorem

In this section we investigate computable versions of the Closed Graph Theorem, which relates properties of an operator $T : X \to Y$ with properties of its graph $\mathrm{graph}(T) := \{(x,y) \in X \times Y : Tx = y\}$.

Theorem 8.1 (Closed Graph Theorem) *Let X, Y be Banach spaces and let $T : X \to Y$ be a linear operator. If* $\mathrm{graph}(T) \subseteq X \times Y$ *is closed, then T is bounded.*

Following the classical proof of the Closed Graph Theorem we immediately obtain a computable version. The proof will mainly be based on Propositions 3.10 and 3.11 on subspaces and product spaces.

Theorem 8.2 (Computable Closed Graph Theorem) *Let X, Y be computable Banach spaces and let $T : X \to Y$ be a linear operator. If* $\mathrm{graph}(T) \subseteq X \times Y$ *is r.e. closed, then $T : X \to Y$ is computable.*

Proof. If X, Y are computable Banach spaces, then $X \times Y$ is a computable Banach space by Proposition 3.11 too. If $\mathrm{graph}(T) \subseteq X \times Y$ is an r.e. closed subset of this product space, then $\mathrm{graph}(T)$ can be considered as computable subspace of $X \times Y$ by Proposition 3.10, which is linear since T is linear. The function $S : \mathrm{graph}(T) \to X, (x, y) \mapsto x$ is obviously linear and bijective and it is computable, since it can be represented as $S = \mathrm{pr}_1 \circ i$ with the projection $\mathrm{pr}_1 : X \times Y \to X$ and the injection $i : \mathrm{graph}(T) \to X \times Y$. Hence, $S^{-1} : X \to \mathrm{graph}(T)$ is computable by the computable Inverse Mapping Theorem 6.3. Thus, the operator $T = \mathrm{pr}_2 \circ i \circ S^{-1}$ is computable too, since the projection $\mathrm{pr}_2 : X \times Y \to Y$ is computable. □

The reader should notice that we have effectivized the standard proof of the Closed Graph Theorem to obtain the computable version. In contrast to this, the computable versions of the Open Mapping Theorem and the Inverse Mapping Theorem have been obtained as consequences of the classical theorems.

Analogously to the Open Mapping Theorem the question appears whether a uniform version of the computable Closed Graph Theorem can be proved, which shows that the operation $\mathrm{graph} :\subseteq \mathcal{C}(X, Y) \to \mathcal{A}_<(X \times Y)$, defined for linear and bounded operators $T : X \to Y$, has a computable inverse. Unfortunately, this is not the case, which can be proved using our standard counterexample. Therefore, we first prove a positive result, which is based on Proposition 4.5 and 4.4.

Theorem 8.3 (Graph Theorem) *Let X, Y be computable metric spaces. The mapping*

$$\mathrm{graph} : \mathcal{C}(X, Y) \to \mathcal{A}(X \times Y), T \mapsto \mathrm{graph}(T)$$

is computable.

Proof. We consider the computable metric spaces (X, d, α), (Y, d', β). Then, $\alpha : \mathbb{N} \to X$ is a computable function such that $\overline{\mathrm{range}(\alpha)} = X$. If $T : X \to Y$ is continuous, then $S : X \to X \times Y, x \mapsto (x, Tx)$ is continuous too and because of continuity of S we obtain that $\mathrm{range}(S\alpha)$ is dense in $\mathrm{range}(S) = \mathrm{graph}(T)$. Using evaluation and type conversion as well as Proposition 4.5, it follows that $\mathrm{graph} : \mathcal{C}(X, Y) \to \mathcal{A}_<(X \times Y)$ is computable.

Now $U : X \times Y \to \mathbb{R}, (x, y) \mapsto d'(Tx, y)$ is continuous, since $T : X \to Y$ and the metric $d' : Y \times Y \to \mathbb{R}$ are continuous and we obtain

$$U(x, y) = 0 \iff d'(Tx, y) = 0 \iff Tx = y \iff (x, y) \in \mathrm{graph}(T).$$

Thus $U^{-1}\{0\} = \mathrm{graph}(T)$ and evaluation together with type conversion and Proposition 4.4 allow to conclude that $\mathrm{graph} : \mathcal{C}(X, Y) \to \mathcal{A}_>(X \times Y)$ is computable. Altogether, this shows that graph : $\mathcal{C}(X,Y) \to \mathcal{A}(X \times Y)$ is computable. $\square$

Unfortunately, the inverse graph^{-1} is not continuous in general, even not for linear bounded and bijective operators $T : X \to Y$ and computable Banach or Hilbert spaces X, Y. As a preparation we prove the following lemma.

Lemma 8.4 *Let X, Y be computable metric spaces. The "swap map"*

$$S : \mathcal{A}(X \times Y) \to \mathcal{A}(Y \times X), A \mapsto \{(y,x) \in Y \times X : (x,y) \in A\}$$

is computable.

The proof follows from the obvious fact that the pointwise "swap function" $s : X \times Y \to Y \times X, (x,y) \mapsto (y,x)$ operates on centers of balls, i.e.

$$sB((x,y),r) = B((y,x),r) = B(s(x,y),r)$$

for all $(x,y) \in X \times Y$ and $r > 0$, provided that $X \times Y$ and $Y \times X$ are endowed with the product space structure according to Proposition 3.4.

Theorem 8.5 *Let $p \geq 1$ be a computable real number. The mapping*

$$\mathrm{graph}^{-1} :\subseteq \mathcal{A}(\ell_p \times \ell_p) \to \mathcal{C}(\ell_p, \ell_p),$$

defined for all closed subsets $A \subseteq \ell_p \times \ell_p$ such that $A = \mathrm{graph}(T)$ for some linear bounded and bijective operator $T : \ell_p \to \ell_p$ with $||T|| = 1$, is not continuous.

Proof. Let us assume that $\mathrm{graph}^{-1} :\subseteq \mathcal{A}(\ell_p \times \ell_p) \to \mathcal{C}(\ell_p, \ell_p)$ would be continuous in the stated sense. By the previous lemma and the Graph Theorem 8.3 it follows that the inversion $I :\subseteq \mathcal{C}(\ell_p, \ell_p) \to \mathcal{C}(\ell_p, \ell_p), T \mapsto T^{-1}$, restricted to linear bounded and bijective operators $T : \ell_p \to \ell_p$ with $||T|| = 1$, would be continuous too, since $I = \mathrm{graph}^{-1} \circ S \circ \mathrm{graph}$. This contradicts Corollary 6.5. $\square$

Using the sequence $(T_n^{-1})_{n\in\mathbb{N}}$ from Corollary 6.7 and the Graph Theorem 8.3, we can formulate the following computable counterexample for the uniform computable version of the Closed Graph Theorem.

Corollary 8.6 *For any computable real number $p \geq 1$ there exists a sequence $(T_n)_{n\in\mathbb{N}}$ of computable linear and bijective operators $T_n : \ell_p \to \ell_p$ which is not computable in $\mathcal{C}(\ell_p, \ell_p)$ but such that $(\mathrm{graph}(T_n))_{n\in\mathbb{N}}$ is a computable sequence of recursive closed subsets of $\ell_p \times \ell_p$.*

A combination of the computable Closed Graph Theorem 8.2 with the Graph Theorem 8.3 yields the following characterization of computable linear operators.

Theorem 8.7 (Computable Linear Operators) *Let X, Y be computable Banach spaces, let $(e_i)_{i\in\mathbb{N}}$ be a computable sequence in X whose linear span is dense in X and let $T : X \to Y$ be a linear operator. Then the following conditions are equivalent:*

(1) $T : X \to Y$ *is computable,*

(2) $T : X \to Y$ *is bounded and maps computable sequences in X to computable sequences in Y,*

(3) $T : X \to Y$ *is bounded and $(Te_k)_{k\in\mathbb{N}}$ is a computable sequence in Y,*

(4) $\mathrm{graph}(T)$ *is an r.e. closed subset of $X \times Y$,*

(5) $\mathrm{graph}(T)$ *is a recursive closed subset of $X \times Y$.*

Proof. "(1)$\Longrightarrow$(2)" Let $T : X \to Y$ be computable. Then T especially is continuous and for any computable sequence $f : \mathbb{N} \to X$, the sequence $T \circ f : \mathbb{N} \to Y$ is computable too, since the composition of computable functions is computable.

"(2)$\Longrightarrow$(3)" Follows directly since (3) is a special case of (2).

"(3)$\Longrightarrow$(4)" Let $\alpha_e\langle k, \langle n_0, ..., n_k\rangle\rangle := \sum_{i=0}^{k} \alpha_{\mathbb{F}}(n_i)e_i$. Then $\alpha_e : \mathbb{N} \to X$ is computable and by linearity of T it follows that $T\alpha_e$ is computable since

$$T\alpha_e\langle k, \langle n_0, ..., n_k\rangle\rangle = \sum_{i=0}^{k} \alpha_{\mathbb{F}}(n_i)Te_i$$

and Te is computable. Moreover, $\mathrm{range}(\alpha_e)$ is dense in X and thus the sequence $f : \mathbb{N} \to X \times Y, i \mapsto (\alpha_e(i), T\alpha_e(i))$ is computable and dense in $\mathrm{graph}(T)$ by continuity of T. Thus $\mathrm{graph}(T)$ is an r.e. closed subset of $X \times Y$.

"(4)$\Longrightarrow$(1)" This follows from the computable closed Graph Theorem 8.2.

"(1)$\Longrightarrow$(5)" This follows from the Graph Theorem 8.3.

"(5)$\Longrightarrow$(4)" Any recursive closed set is r.e. closed. □

The equivalence of (1), (2) and (3) can also be proved directly without using the Closed Graph Theorem (cf. Corollary 4.4.50 in [Bra99b]). It is also easy to prove that any bounded linear operator T with an r.e. closed graph is computable. Thus, the computable Closed Graph Theorem 8.2 could be obtained directly as a corollary of the classical Closed Graph Theorem (without using the computable Open Mapping Theorem). Can we also derive the computable Open Mapping Theorem 5.4 from the computable Closed Graph Theorem 8.2? At least the classical proof, using the quotient space $X_T := X/\mathrm{kern}(T)$ cannot be used for this purpose. The problem is that the quotient norm, defined by

$$||x + \mathrm{kern}(T)||_T := \inf_{z \in \mathrm{kern}(T)} ||x - z||$$

is computable, if and only if the distance function $d_{\mathrm{kern}(T)} : X \to \mathbb{R}$ is computable. But this is only the case for some computable linear operators $T : X \to Y$ and not in general, as the following counterexample shows.

Lemma 8.8 *For any computable real number $p \geq 1$ there exists a linear computable operator $T : \ell_p \to \ell_p$ such that* $\mathrm{kern}(T)$ *is a co-r.e. closed subset of ℓ_p which is not r.e. closed.*

Proof. Let $f : \mathbb{N} \to \mathbb{N}$ be some computable function such that range(f) is not recursive. We define a computable sequence $a : \mathbb{N} \to \mathbb{R}$ by

$$a_k := \begin{cases} 0 & \text{if } k \notin \mathrm{range}(f) \\ 2^{-m} & \text{if } m = \min\{n : f(n) = k\} \end{cases}$$

and a computable operator $T : \ell_p \to \ell_p$ by $T(x_k)_{k\in\mathbb{N}} := (a_k x_k)_{k\in\mathbb{N}}$. Obviously, T is linear and $\mathrm{kern}(T) = T^{-1}\{0\} = \{(x_k)_{k\in\mathbb{N}} \in \ell_p : (\forall i)(a_i \neq 0 \Longrightarrow x_i = 0)\}$. If we define $f : \ell_p \to \mathbb{R}$ by $f(x) := ||Tx||$, then f is computable and $\mathrm{kern}(T) = f^{-1}\{0\}$ is co-r.e. closed by Proposition 4.4. Now let $(e_i)_{i\in\mathbb{N}}$ be the computable sequence of unit vectors $e_i \in \ell_p$, as defined in Proposition 3.8(2). Then

$$e_i \in \mathrm{kern}(T) \iff a_i = 0 \iff i \notin \mathrm{range}(f).$$

If the distance function $d_{\mathrm{kern}(T)} : \ell_p \to \mathbb{R}_>, x \mapsto \inf_{z\in\mathrm{kern}(T)} ||x - z||$ of $\mathrm{kern}(T)$ would be upper semi-computable, then $\mathbb{N} \setminus \mathrm{range}(f)$ would be r.e. since

$$d_{\mathrm{kern}(T)}(e_i) < 1 \iff i \notin \mathrm{range}(f).$$

Contradiction! Thus, $d_{\mathrm{kern}(T)}$ is not upper semi-computable and thus $\mathrm{kern}(T)$ is not r.e. closed (cf. Corollary 3.14 in [BP00]). □

9 The Uniform Boundedness Theorem

In this section we will study computable versions of the Uniform Boundedness Theorem. The theorem states that each pointwise bounded sequence of bounded linear operators is also uniformly bounded, more precisely:

Theorem 9.1 (Uniform Boundedness Theorem) *Let X be a Banach space, Y a normed space and let $(T_i)_{i\in\mathbb{N}}$ be a sequence of bounded linear operators $T_i : X \to Y$. If $\{||T_i x|| : i \in \mathbb{N}\}$ is bounded for each $x \in X$, then $\{||T_i|| : i \in \mathbb{N}\}$ is bounded.*

A proof of the theorem can be found in [GP65]. We start with an investigation of the bound $||T|| := \sup_{x\in\overline{B}(0,1)} ||Tx||$ of operators $T : X \to Y$. As a first observation we note that the closed unit ball $\overline{B}(0,1)$ of a computable normed space X is a recursive closed subset.

Lemma 9.2 *The closed unit ball $\overline{B}(0,1)$ of a computable normed space X is a recursive closed subset of X.*

Proof. We prove

$$d_{\overline{B}(0,1)}(x) = \max\{||x|| - 1, 0\}. \tag{2}$$

Since the norm is computable, it follows that $d_{\overline{B}(0,1)} : X \to \mathbb{R}$ is a computable function. By Proposition 4.6 we obtain that $\overline{B}(0,1)$ is a recursive closed subset of X. For the proof of Equation (2) let $x \in X$. If $||x|| < 1$, i.e. $x \in B(0,1)$, then $d_{\overline{B}(0,1)}(x) = 0 = \max\{||x|| - 1, 0\}$ follows. Thus, let $||x|| \geq 1$. Then $||\frac{x}{||x||}|| = 1$ and thus $y := \frac{x}{||x||} \in \overline{B}(0,1)$. We obtain

$$d_{\overline{B}(0,1)}(x) \leq ||x - y|| = \left|1 - \frac{1}{||x||}\right| \cdot ||x|| = ||x|| - 1 = \max\{||x|| - 1, 0\}.$$

Now let $y \in \overline{B}(0,1)$ be some arbitrary point. Then $||x - y|| \geq ||x|| - ||y|| \geq ||x|| - 1$ and hence

$$d_{\overline{B}(0,1)}(x) = \inf_{y\in\overline{B}(0,1)} ||x - y|| \geq ||x|| - 1 = \max\{||x|| - 1, 0\}.$$

Altogether, this proves Equation (2). $\square$

As a consequence we obtain that the bound $||T||$ can be computed from below, if it exists.

Theorem 9.3 (Semi-computable bound) *Let X, Y be computable normed spaces. The partial map*

$$|| \, || :\subseteq \mathcal{C}(X,Y) \to \mathbb{R}_<, T \mapsto ||T||$$

is computable.

Proof. By the previous Lemma 9.2 the closed unit ball $\overline{B}(0,1)$ is a recursive closed subset and hence especially an r.e. closed subset. Thus, there exists some computable sequence $f : \mathbb{N} \to X$ such that $\overline{\mathrm{range}(f)}$ is dense in $\overline{B}(0,1)$. By continuity it follows

$$||T|| = \sup_{x \in \overline{B}(0,1)} ||Tx|| = \sup_{n \in \mathbb{N}} ||Tf(n)||$$

for all bounded continuous operators $T : X \to Y$. Since the norm of Y is computable and $\sup :\subseteq \mathbb{R}^{\mathbb{N}} \to \mathbb{R}_<$ is computable, we obtain the desired result with help of the evaluation property. □

As a corollary we immediately obtain that the bound of any computable bounded operator is a left-computable real number.

Corollary 9.4 *Let X, Y be computable normed spaces. If $T : X \to Y$ is a computable linear operator, then $||T||$ is a left-computable real number.*

On the other hand, we can conclude from Proposition 5.7 and Corollary 6.3 that the bound of a computable linear operator is not necessarily computable: for any right-computable but not left-computable real number $a \in (0,1]$ there exists some computable diagonal operator $T_a : \ell_p \to \ell_p$ with $||T_a^{-1}|| = \frac{1}{a}$ by Proposition 5.7. Thus, $T := T_a^{-1}$ is computable by Corollary 6.3, but $||T|| = \frac{1}{a}$ is not right-computable.

Corollary 9.5 *For any computable real number $p \geq 1$ there exists some computable linear operator $T : \ell_p \to \ell_p$ such that $||T||$ is not right-computable.*

Next we want to show that the map $T \mapsto ||T||$ is also discontinuous. Therefore we formulate the following proposition which corresponds to Proposition 5.7 (but the reader should notice that $\mathcal{B}(\ell_p, \ell_p)$ is replaced by $\mathcal{C}(\ell_p, \ell_p)$).

Proposition 9.6 (Inverse diagonal operator) *Let $p \geq 1$ be a computable real. There exists a computable multi-valued operation $\sigma :\subseteq \mathbb{R}_> \rightrightarrows \mathcal{C}(\ell_p, \ell_p)$ such that for any $a \in \mathbb{R}_>$ with $a \in [\frac{1}{2}, 1]$ there exists some $T \in \sigma(a)$ and all such $T : \ell_p \to \ell_p$ have as inverse T^{-1} some diagonal operator of a.*

Proof. Given a real number $a \in \mathbb{R}_>$ with $a \in [\frac{1}{2}, 1]$, we can effectively find a decreasing sequence $(a_n)_{n\in\mathbb{N}}$ of rational numbers $a_n \in \mathbb{Q}$ such that $a_0 = 1$ and $a = \inf_{n\in\mathbb{N}} a_n$. We define a diagonal operator $T_a : \ell_p \to \ell_p$ of a by $T_a(x_k)_{k\in\mathbb{N}} := (a_k x_k)_{k\in\mathbb{N}}$ for all $(x_k)_{k\in\mathbb{N}} \in \ell_p$. Given some $x = (x_k)_{k\in\mathbb{N}}$ and a precision $m \in \mathbb{N}$ we can effectively find some $n \in \mathbb{N}$ and numbers $q_0, ..., q_n \in Q_{\mathbb{F}}$ such that $||\sum_{i=0}^n q_i e_i - x||_p < 2^{-m-1}$. It follows $||T_a^{-1}|| \le \frac{1}{a} \le 2$ and hence

$$\left|\left|T_a^{-1}\left(\sum_{i=0}^n q_i e_i\right) - T_a^{-1}(x)\right|\right|_p \le ||T_a^{-1}|| \cdot \left|\left|\sum_{i=0}^n q_i e_i - x\right|\right|_p < 2^{-m}$$

By linearity of T_a^{-1} we obtain $T_a^{-1}(\sum_{i=0}^n q_i e_i) = \sum_{i=0}^n q_i T_a^{-1}(e_i) = \sum_{i=0}^n \frac{q_i}{a_i}$ and thus we can evaluate $T := T_a^{-1}$ effectively up to any given precision m. Using type conversion we can actually prove that there exists an operation σ with the desired properties. $\square$

As a consequence we obtain that $T \mapsto ||T||$ is not continuous in general.

Theorem 9.7 *Let $p \ge 1$ be a computable real. The mapping $T \mapsto ||T||$, defined for linear bounded and bijective operators $T : \ell_p \to \ell_p$, is not continuous.*

Proof. Let us assume that $||\ || : \mathcal{C}(\ell_p, \ell_p) \to \mathbb{R}, T \mapsto ||T||$ is continuous. Then the composition $||\ || \circ \sigma :\subseteq \mathbb{R}_> \to \mathbb{R}, a \mapsto \frac{1}{a}$ would be continuous too by the previous Proposition 9.6 and Lemma 5.6. Since $I :\subseteq \mathbb{R} \to \mathbb{R}, x \mapsto \frac{1}{x}$ is continuous, it follows that the identity $\mathrm{id}_A :\subseteq \mathbb{R}_> \to \mathbb{R}$ is continuous on $A := [\frac{1}{2}, 1]$ (and thus $\rho_>|^A \le_t \rho$), which is obviously wrong. Contradiction! $\square$

Using Theorem 9.3 and the fact that $\sup :\subseteq \mathbb{R}_<^{\mathbb{N}} \to \mathbb{R}_<$ is computable, we obtain the following (not very surprising) computable version of the Uniform Boundedness Theorem which states that we can compute the uniform bound of a pointwise bounded sequence of linear bounded operators from below.

Corollary 9.8 (Semi-computable Uniform Boundedness Theorem) *Let X be a computable Banach space and let Y be a computable normed space. Then the mapping*

$$S :\subseteq \mathcal{C}(X, Y)^{\mathbb{N}} \to \mathbb{R}_<, (T_i)_{i\in\mathbb{N}} \mapsto \sup_{i\in\mathbb{N}} ||T_i||$$

with $\mathrm{dom}(S) = \{(T_i)_{i\in\mathbb{N}} : T_i : X \to Y$ *linear and bounded and* $\{||T_i x|| : i \in \mathbb{N}\}$ *is bounded for each* $x \in X\}$ *is computable.*

Here, the classical Uniform Boundedness Theorem guarantees that the mapping S is well-defined. We obtain directly the following corollary.

Corollary 9.9 *Let X be a computable Banach space and let Y be a computable normed space. If $(T_i)_{i\in\mathbb{N}}$ is a computable sequence of computable bounded operators $T_i : X \to Y$ such that $\{||T_i x|| : i \in \mathbb{N}\}$ is bounded for each $x \in X$, then $\sup_{i\in\mathbb{N}} ||T_i||$ is a left-computable real number.*

On the other hand, using the constant sequence with the computable operator $T : \ell_p \to \ell_p$ from Corollary 9.5, one can see that the uniform bound is not computable in general. Correspondingly, one can use Theorem 9.7 to show that the uniform bound does not continuously depend on the sequence of operators.

Up to now we have only seen that, given an operator T, we can compute arbitrary lower rational bounds of $||T||$ which is not very helpful. If we cannot compute $||T||$ precisely, then it would be useful to compute at least *some* upper bound $s \geq ||T||$. And actually this is possible, as the following result shows.

Theorem 9.10 (Bound) *Let X, Y be computable normed spaces. There exists a computable multi-valued operation $N :\subseteq \mathcal{C}(X,Y) \rightrightarrows \mathbb{R}$, such that for any linear bounded operator $T : X \to Y$, there exists some $s \in N(T)$ and $||T|| \leq s$ holds for all such s.*

Proof. Given a linear bounded operator $T : X \to Y$, we can effectively compute $T^{-1}B(0,1) \in \mathcal{O}(X)$ by Theorem 6.4 and the evaluation property since $B(0,1)$ is a computable point in $\mathcal{O}(X)$. By linearity of T, we obtain $0 \in T^{-1}B(0,1)$ and thus we can effectively find some rational number $r > 0$ such that $B(0,r) \subseteq T^{-1}B(0,1)$ by Lemma 5.2. By Lemma 6.8 we obtain $||T|| \leq \frac{1}{r}$. Thus, $s := \frac{1}{r}$ is an appropriate result. $\square$

Now the question appears whether we can prove a corresponding computable version of the Uniform Boundedness Theorems which allows to compute *some* upper bound of the uniform bound. Actually, the question is: which input information on a sequence $(T_i)_{i\in\mathbb{N}}$ of linear bounded and pointwise bounded operators T_i suffices to compute some upper bound on $\sup_{i\in\mathbb{N}} ||T_i||$? The following theorem shows that the pure knowledge of pointwise boundedness does not suffice, not even in case of Euclidean space.

Theorem 9.11 *There exists no continuous operation $\beta :\subseteq \mathcal{C}(\mathbb{R})^{\mathbb{N}} \rightrightarrows \mathbb{R}$ with the following property: for all sequences $(T_i)_{i\in\mathbb{N}}$ of linear bounded operators $T_i : \mathbb{R} \to \mathbb{R}$ such that $\{||T_i x|| : i \in \mathbb{N}\}$ is bounded for each $x \in \mathbb{R}$, there exists some $s \in \beta(T_i)_{i\in\mathbb{N}}$ and $\sup_{i\in\mathbb{N}} ||T_i|| \leq s$ for all such s.*

Proof. There exists a computable operation $A :\subseteq \mathbb{R}_< \rightrightarrows \mathbb{R}^{\mathbb{N}}$ such that $[1,\infty) \subseteq \mathrm{dom}(A)$ and $(a_n)_{n\in\mathbb{N}} \in A(a)$ implies $a_0 = 1$ and $\sup_{n\in\mathbb{N}} a_n = a$

for each $a \in [1, \infty)$. The function $B : \mathbb{R}^{\mathbb{N}} \to \mathcal{C}(\mathbb{R})^{\mathbb{N}}, (a_n)_{n\in\mathbb{N}} \mapsto (x \mapsto a_n x)_{n\in\mathbb{N}}$ is computable too, as one can show by evaluation and type conversion. Now let us assume that there exists a continuous operation $\beta :\subseteq \mathcal{C}(\mathbb{R})^{\mathbb{N}} \rightrightarrows \mathbb{R}$ with the property described above. Then $F := \beta \circ B \circ A :\subseteq \mathbb{R}_< \rightrightarrows \mathbb{R}$ is a continuous operation such that $[1, \infty) \subseteq \mathrm{dom}(F)$ and $s \in F(a)$ implies $s \geq a$ for all $a \in \mathbb{R}_<$ with $a \geq 1$. But such a continuous operation can obviously not exist. Contradiction! □

The following result shows that it does not help if the pointwise bounds on some fundamental sequence of unit vectors are given as additional input information. The proof uses a similar idea as the proof of Proposition 9.6.

Theorem 9.12 *Let $p \geq 1$ be some computable real number. There exists no continuous operation $\beta :\subseteq \mathcal{C}(\ell_p, \ell_p)^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \rightrightarrows \mathbb{R}$ with the following property: for all sequences $(s_j)_{j\in\mathbb{N}}$ of real numbers and for all sequences $(T_i)_{i\in\mathbb{N}}$ of linear bounded operators $T_i : \mathbb{R} \to \mathbb{R}$ such that $\{||T_i x|| : i \in \mathbb{N}\}$ is bounded for each $x \in \mathbb{R}$ and $\sup_{i\in\mathbb{N}} ||T_i e_j|| = s_j$, there exists some $s \in \beta((T_i)_{i\in\mathbb{N}}, (s_j)_{j\in\mathbb{N}})$ and $\sup_{i\in\mathbb{N}} ||T_i|| \leq s$ for all such s.*

Proof. Let us assume that a continuous operation $\beta :\subseteq \mathcal{C}(\ell_p, \ell_p)^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \rightrightarrows \mathbb{R}$ with the stated property exists. Given a real number $a \in \mathbb{R}_<$ with $a \in [1, \infty)$, we can effectively find an increasing sequence $(a_n)_{n\in\mathbb{N}}$ of rational numbers $a_n \in \mathbb{Q}$ such that $a_0 = 1$ and $a = \sup_{n\in\mathbb{N}} a_n$. Let

$$a_{ij} := \begin{cases} a_j & \text{for } j = 0, ..., i \\ a_i & \text{for } j > i \end{cases}$$

We define operators $T_i : \ell_p \to \ell_p$ by $T_i(x_k)_{k\in\mathbb{N}} := (a_{ik} x_k)_{k\in\mathbb{N}}$ for all $(x_k)_{k\in\mathbb{N}} \in \ell_p$. Then $||T_i|| = a_i$ and $\sup_{i\in\mathbb{N}} ||T_i e_j|| = a_j$. Given some $x = (x_k)_{k\in\mathbb{N}}$, $i \in \mathbb{N}$ and a precision $m \in \mathbb{N}$ we can effectively find some $n \in \mathbb{N}$ and numbers $q_0, ..., q_n \in Q_{\mathbb{F}}$ such that $|| \sum_{j=0}^{n} q_j e_j - x||_p < \frac{1}{a_i} 2^{-m}$ and hence

$$\left|\left| T_i\left(\sum_{j=0}^{n} q_j e_j\right) - T_i(x) \right|\right|_p \leq ||T_i|| \cdot \left|\left| \sum_{j=0}^{n} q_j e_j - x \right|\right|_p < 2^{-m}$$

By linearity of T_i we obtain $T_i(\sum_{j=0}^{n} q_j e_j) = \sum_{j=0}^{n} q_j T_i(e_j) = \sum_{j=0}^{n} q_j a_{ij}$ and thus we can evaluate each T_i effectively up to any given precision m. Using type conversion we can actually prove that there exists an operation $\sigma :\subseteq \mathbb{R}_< \rightrightarrows \mathcal{C}(\ell_p, \ell_p)^{\mathbb{N}}$ that maps each a to a sequence $(T_i)_{i\in\mathbb{N}}$ of operators as described above. Now by assumption we can continuously find some $s \in \beta((T_i)_{i\in\mathbb{N}}, (a_j)_{j\in\mathbb{N}})$ such that $s \geq \sup_{i\in\mathbb{N}} ||T_i|| = \sup_{i\in\mathbb{N}} a_i = a$. Altogether, we have proved that under the assumption there is a continuous operation

$\gamma :\subseteq \mathbb{R}_< \rightrightarrows \mathbb{R}$ which maps each $a \in \mathbb{R}_<$ with $a \geq 1$ to some $s \in \mathbb{R}$ with $s > a$. But such a continuous operation can obviously not exist. Contradiction! □

Although it does not help to know the pointwise bounds on some fundamental sequence of unit vectors, it is sufficient to have *all* pointwise bounds as additional input information, as the following result shows. Therefore, we will consider the input as operator $T : X \to \mathcal{B}(\mathbb{N}, Y)$ and for all such operators we define $T_i : X \to Y, x \mapsto T(x)(i)$. It is clear that an operator T is well-defined, if and only if $(T_i)_{i\in\mathbb{N}}$ is pointwise bounded and T is linear, if and only if all T_i are linear. By the classical Uniform Boundedness Theorem a linear operator T is bounded, if and only if all T_i are bounded and the sequence $(T_i)_{i\in\mathbb{N}}$ is pointwise bounded. In this case we obtain

$$||T|| = \sup_{x\in\overline{B}(0,1)} ||Tx|| = \sup_{x\in\overline{B}(0,1)} \sup_{i\in\mathbb{N}} ||T_i x|| = \sup_{i\in\mathbb{N}} \sup_{x\in\overline{B}(0,1)} ||T_i x|| = \sup_{i\in\mathbb{N}} ||T_i||,$$

thus, the uniform bound of $(T_i)_{i\in\mathbb{N}}$ is nothing but the bound of T. The main technical difficulty in the proof of the following theorem is the fact that $\mathcal{B}(\mathbb{N}, Y)$ is a non-separable Banach space and therefore we cannot derive the result directly from Theorem 9.10.

Theorem 9.13 (Computable Uniform Boundedness Theorem) *Let X be a computable Banach space and let Y be a computable normed space. There exists a computable operation $\beta :\subseteq \mathcal{C}(X, \mathcal{B}(\mathbb{N}, Y)) \rightrightarrows \mathbb{R}$ with the following property: for all linear bounded operators $T : X \to \mathcal{B}(\mathbb{N}, Y)$ there exists some $s \in \beta(T)$ and $||T|| = \sup_{i\in\mathbb{N}} ||T_i|| \leq s$ for all such s.*

Proof. First of all, we prove that the operation

$$U : \mathcal{C}(X, \mathcal{B}(\mathbb{N}, Y)) \to \mathcal{O}(X), T \mapsto T^{-1}B(0,1)$$

is computable. Therefore, consider

$$f : \mathcal{B}(\mathbb{N}, Y) \to \mathbb{R}, (y_i)_{i\in\mathbb{N}} \mapsto \max\{1 - ||(y_i)_{i\in\mathbb{N}}||, 0\}.$$

Since the norm $||\ || : \mathcal{B}(\mathbb{N}, Y) \to \mathbb{R}$ is computable, it follows that f is computable too. Thus, given $T \in \mathcal{C}(X, \mathcal{B}(\mathbb{N}, Y))$ we can effectively find $g := f \circ T : X \to \mathbb{R}$, using evaluation and type conversion. Now

$$g^{-1}\{0\} = T^{-1}f^{-1}\{0\} = T^{-1}(\mathcal{B}(\mathbb{N}, Y) \setminus B(0,1))$$

and thus we can determine $U(T) = T^{-1}B(0,1) = X \setminus g^{-1}\{0\} \in \mathcal{O}(X)$ effectively by Proposition 4.4.

By linearity of T, we obtain $0 \in T^{-1}B(0,1)$ and thus we can effectively find some rational number $r > 0$ such that $B(0,r) \subseteq U(T) = T^{-1}B(0,1)$ by Lemma 5.2. By Lemma 6.8 we obtain $||T|| \leq \frac{1}{r}$. Thus, $s := \frac{1}{r}$ is an appropriate result of $\beta(T)$. $\square$

Can we also effectivize the contraposition of the formulation of the Uniform Boundedness Theorem? Thus, given a sequence of linear bounded operators $T_i : X \to Y$ such that $\{||T_i|| : i \in \mathbb{N}\}$ is unbounded, can we effectively find some witness $x \in X$ such that $\{||T_i x|| : i \in \mathbb{N}\}$ is unbounded? The answer is "yes", as the following theorem shows. The proof is a direct effectivization of the classical proof of the Uniform Boundedness Theorem [GP65] and it uses the computable Baire Category Theorem B.4 from Appendix B.

Theorem 9.14 (Contra computable Uniform Boundedness Theorem) *Let X be a computable Banach space and let Y be a computable normed space. There exists a computable multi-valued operation $\beta' :\subseteq \mathcal{C}(X,Y)^{\mathbb{N}} \rightrightarrows X^{\mathbb{N}}$ with the following property: for all sequences $(T_i)_{i\in\mathbb{N}}$ of linear and bounded operators $T_i : X \to Y$ such that $\{||T_i|| : i \in \mathbb{N}\}$ is unbounded, there exists a sequence $(x_n)_{n\in\mathbb{N}} \in \beta'(T_i)_{i\in\mathbb{N}}$ and all such sequences are dense in the set $\{x \in X : \{||T_i x|| : i \in \mathbb{N}\}$ is unbounded$\}$.*

Proof. Let $(T_i)_{i\in\mathbb{N}}$ be a sequence of linear bounded operators $T_i : X \to Y$ such that $\{||T_i|| : i \in \mathbb{N}\}$ is unbounded. Let $f_i : X \to \mathbb{R}$ be defined by $f_i(x) := ||T_i x||$ and

$$A_n := \{x \in X : (\forall i \in \mathbb{N})\ ||T_i x|| \leq n\} = \bigcap_{i=0}^{\infty} f_i^{-1}[0,n].$$

Thus, using the fact that the norm $||\ || : Y \to \mathbb{R}$ is computable, by Theorem 6.4 and Proposition B.9(1) from Appendix B we obtain that the mapping $\alpha : \mathcal{C}(X,Y)^{\mathbb{N}} \to \mathcal{A}_>(X)^{\mathbb{N}}, (T_i)_{i\in\mathbb{N}} \to (A_n)_{n\in\mathbb{N}}$ is computable, since $([0,n])_{n\in\mathbb{N}}$ is a computable sequence in $\mathcal{A}_>(\mathbb{R})$. Moreover, we obtain

$$\begin{aligned}\bigcup_{n=0}^{\infty} A_n &= \{x \in X : (\exists n \in \mathbb{N})(\forall i \in \mathbb{N})\ ||T_i x|| \leq n\} \\ &= \{x \in X : \{||T_i x|| : i \in \mathbb{N}\} \text{ bounded}\}.\end{aligned}$$

Now, let us assume that some set A_n is somewhere dense, i.e. there exists some $x \in X$ and some $\varepsilon > 0$ such that $B(x,\varepsilon) \subseteq A_n$. Then there is some $r \in \mathbb{N}$ such that

$$\overline{B}(0,1) \subseteq rB(x,\varepsilon) \subseteq rA_n \subseteq A_{rn}.$$

Thus, $||T_i|| = \sup_{x\in\overline{B}(0,1)} ||T_ix|| \leq rn$ for all $i \in \mathbb{N}$ which contradicts the assumption that $\{||T_i|| : i \in \mathbb{N}\}$ is unbounded. Hence, $(A_n)_{n\in\mathbb{N}}$ is a sequence of nowhere dense sets. By the Computable Baire Category Theorem B.4 from Appendix B there exists a computable operation $\Delta :\subseteq \mathcal{A}_>(X)^{\mathbb{N}} \rightrightarrows X^{\mathbb{N}}$ such that there exists some $(x_n)_{n\in\mathbb{N}} \in \Delta(A_n)_{n\in\mathbb{N}}$ whenever $(A_n)_{n\in\mathbb{N}}$ is a sequence of nowhere dense closed subsets $A_n \subseteq X$ and $(x_n)_{n\in\mathbb{N}}$ is dense in $X \setminus \bigcup_{n=0}^{\infty} A_n$ for all such sequences $(x_n)_{n\in\mathbb{N}}$. Thus, $\beta' := \Delta \circ \alpha$ is a computable operation with the desired properties. □

We obtain the following weaker non-uniform corollary.

Corollary 9.15 *Let X be a computable Banach space and let Y be a computable normed space. For any computable sequences $(T_i)_{i\in\mathbb{N}}$ of linear and computable operators $T_i : X \to Y$ such that $\{||T_i|| : i \in \mathbb{N}\}$ is unbounded, there exists a computable sequence $(x_n)_{n\in\mathbb{N}}$ in X which is dense in the set $\{x \in X : \{||T_ix|| : i \in \mathbb{N}\}$ is unbounded$\}$.*

10 Condensation of Singularities

In this section we want to study a computable version of Banach's Theorem on Condensation of Singularities.

Theorem 10.1 (Condensation of Singularities) *Let X be a Banach space and Y a normed space. If $(T_{mn})_{n\in\mathbb{N}}$ is a sequence of linear and bounded operators $T_{mn} : X \to Y$ such that there exists an $x_m \in X$ for every $m \in \mathbb{N}$ with $\sup_{n\in\mathbb{N}} ||T_{mn}(x_m)|| = \infty$, then there also exists an $x \in X$ such that $\sup_{n\in\mathbb{N}} ||T_{mn}(x)|| = \infty$ for all $m \in \mathbb{N}$.*

A proof can be found in [GP65] and it can be effectivized straightforwardly using the computable Baire Category Theorem B.4 from Appendix B. The proof is very similar to the proof of the contra computable Uniform Boundedness Theorem 9.14.

Theorem 10.2 (Computable Condensation of Singularities) *Let X be a computable Banach space and Y a computable normed space. There exists a computable operation $C :\subseteq \mathcal{C}(X,Y)^{\mathbb{N}} \rightrightarrows X^{\mathbb{N}}$ with the following property: if $(T_i)_{i\in\mathbb{N}}$ is a sequence of linear bounded operators $T_i : X \to Y$ such that there exists an $x_m \in X$ for every $m \in \mathbb{N}$ with $\sup_{n\in\mathbb{N}} ||T_{\langle m,n\rangle}(x_m)|| = \infty$, then there exists a sequence $(y_j)_{j\in\mathbb{N}} \in C(T_i)_{i\in\mathbb{N}}$ and all such sequences $(y_j)_{j\in\mathbb{N}}$ are dense in the set $\{x \in X : (\forall m) \sup_{n\in\mathbb{N}} ||T_{\langle m,n\rangle}(x)|| = \infty\}$.*

Proof. Let $(T_i)_{i\in\mathbb{N}}$ be a sequence of linear bounded operators $T_i : X \to Y$ such that there exists an $x_m \in X$ for every $m \in \mathbb{N}$ with $\sup_{n\in\mathbb{N}} ||T_{\langle m,n\rangle}(x_m)|| = \infty$. Let $f_i : X \to \mathbb{R}$ be defined by $f_{\langle m,n\rangle}(x) := ||T_{\langle m,n\rangle}(x)||$ and

$$A_{\langle m,k\rangle} := \{x \in X : (\forall n)\ ||T_{\langle m,n\rangle}(x)|| \leq k\} = \bigcap_{n=0}^{\infty} f^{-1}_{\langle m,n\rangle}[0,k].$$

Thus, using the fact that the norm $||\ || : Y \to \mathbb{R}$ is computable, by Theorem 6.4 and Proposition B.9(1) from Appendix B we obtain that the mapping $\alpha : \mathcal{C}(X,Y)^{\mathbb{N}} \to \mathcal{A}_>(X)^{\mathbb{N}}, (T_i)_{i\in\mathbb{N}} \to (A_i)_{i\in\mathbb{N}}$ is computable, since $([0,k])_{k\in\mathbb{N}}$ is a computable sequence in $\mathcal{A}_>(\mathbb{R})$. Moreover, we obtain

$$\begin{aligned} \bigcup_{\langle m,k\rangle=0}^{\infty} A_{\langle m,k\rangle} &= \left\{x \in X : (\exists m,k)(\forall n)\ ||T_{\langle m,n\rangle}(x)|| \leq k\right\} \\ &= X \setminus \left\{x \in X : (\forall m)\ \sup_{n\in\mathbb{N}} ||T_{\langle m,n\rangle}(x)|| = \infty\right\}. \end{aligned}$$

Now, let us assume that some set $A_{\langle m,k\rangle}$ is somewhere dense, i.e. there exists some $x \in X$ and some $\varepsilon > 0$ such that $B(x,\varepsilon) \subseteq A_{\langle m,k\rangle}$. Then there is some $r \in \mathbb{N}$ such that

$$x_m \in rB(x,\varepsilon) \subseteq rA_{\langle m,k\rangle} \subseteq A_{\langle m,rk\rangle}.$$

Thus, $||T_{\langle m,n\rangle}(x_m)|| \leq rk$ for all $n \in \mathbb{N}$ but this is a contradiction to the condition $\sup_{n\in\mathbb{N}} ||T_{\langle m,n\rangle}(x_m)|| = \infty$. Hence, $(A_i)_{i\in\mathbb{N}}$ is a sequence of nowhere dense sets. By the Computable Baire Category Theorem B.4 from Appendix B there exists a computable operation $\Delta :\subseteq \mathcal{A}_>(X)^{\mathbb{N}} \rightrightarrows X^{\mathbb{N}}$ such that there exists some $(y_j)_{j\in\mathbb{N}} \in \Delta(A_i)_{i\in\mathbb{N}}$ whenever $(A_i)_{i\in\mathbb{N}}$ is a sequence of nowhere dense closed subsets $A_i \subseteq X$ and $(y_j)_{j\in\mathbb{N}}$ is dense in $X \setminus \bigcup_{i=0}^{\infty} A_i$ for all such sequences $(y_j)_{j\in\mathbb{N}}$. Thus, $C := \Delta \circ \alpha$ is a computable operation with the desired properties. □

We obtain the following weaker non-uniform corollary.

Corollary 10.3 *Let X be a computable Banach space and Y a computable normed space. If $(T_i)_{i\in\mathbb{N}}$ is a computable sequence of linear computable operators $T_i : X \to Y$ such that there exists an $x_m \in X$ for every $m \in \mathbb{N}$ with $\sup_{n\in\mathbb{N}} ||T_{\langle m,n\rangle}(x_m)|| = \infty$, then there exists a computable sequence $(y_j)_{j\in\mathbb{N}}$ in X which is dense in the set $\{x \in X : (\forall m) \sup_{n\in\mathbb{N}} ||T_{\langle m,n\rangle}(x)|| = \infty\}$.*

11 Divergent Fourier Series

One standard example of an application of the Uniform Boundedness Theorem is the construction of a continuous function $f : [0,2\pi] \to \mathbb{R}$ whose Fourier series diverges at $t = 0$ (cf. [GP65]). We can directly transfer this to the computable setting and prove that there exists a computable function f whose Fourier series does not converge at $t = 0$.

Theorem 11.1 *There exists a computable function $f : [0, 2\pi] \to \mathbb{R}$ such that*

$$s_i := \frac{1}{2\pi} \int_0^{2\pi} f(t) \frac{\sin(i + \frac{1}{2})t}{\sin \frac{1}{2}t} \, dt$$

does not converge to $f(0)$ as $i \to \infty$.

Proof. We consider the computable Banach space $\mathcal{C}[0, 2\pi]$ and we define a sequence of operators $T_i : \mathcal{C}[0, 2\pi] \to \mathbb{R}$ by

$$T_i(f) := \frac{1}{2\pi} \int_0^{2\pi} f(t) \frac{\sin(i + \frac{1}{2})t}{\sin \frac{1}{2}t} \, dt$$

One can prove that $||T_i|| = \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\sin(i+1/2)t}{\sin(t/2)} \right| dt$ and thus each T_i is bounded, whereas $\{||T_i|| : i \in \mathbb{N}\}$ is unbounded [GP65]. Using evaluation and type conversion, one can prove that $(T_i)_{i \in \mathbb{N}}$ is a computable sequence of linear and computable operators $T_i : \mathcal{C}[0, 2\pi] \to \mathbb{R}$ in $\mathcal{C}(\mathcal{C}[0, 2\pi])$. This follows from the fact that integration is computable (cf. Theorem 6.4.1.2 in [Wei00]). Now Corollary 9.15 yields a computable sequence $(f_n)_{n \in \mathbb{N}}$ in $\mathcal{C}[0, 2\pi]$ which is dense in the set $\{f \in \mathcal{C}[0, 2\pi] : \{||T_i f|| : i \in \mathbb{N}\}$ is unbounded $\}$. Thus, all functions $f := f_n : [0, 2\pi] \to \mathbb{R}$ are computable and fulfill the desired property. □

Using the computable version of the Theorem on Condensation of Singularities 10.2 we could even prove that, given a sequence of computable numbers $(x_n)_{n \in \mathbb{N}}$ in $[0, 2\pi]$, we can effectively find a computable function $f : [0, 2\pi] \to \mathbb{R}$ such that the Fourier series of f does not converge to f for all x_n. We will not formulate this theorem here.

12 The Banach-Steinhaus Theorem

In this section we discuss a computable version of the Banach-Steinhaus Theorem [DS59].

Theorem 12.1 (Banach-Steinhaus Theorem) *Let X be a Banach space and Y be a normed space and $(T_i)_{i \in \mathbb{N}}$ a sequence of linear and bounded operators $T_i : X \to Y$ which converges pointwise. Then by*

$$Tx := \lim_{n \to \infty} T_n x$$

a linear and bounded operator $T : X \to Y$ is defined.

Additionally, $||T|| \leq \sup_{n\in\mathbb{N}} ||T_n||$ holds in the situation of the theorem. The following simple example shows (similarly as in Theorem 9.11) that a given computable sequence of computable and pointwise converging operators needs not to converge to a computable operator.

Example 12.2 *Let $(a_n)_{n\in\mathbb{N}}$ be an increasing computable sequence of real numbers such that $a := \sup_{n\in\mathbb{N}} a_n$ exists, but is not computable. Then the sequence $(T_i)_{i\in\mathbb{N}}$ of mappings $T_i : \mathbb{R} \to \mathbb{R}, x \mapsto a_i x$ is a computable sequence of computable linear operators such that $Tx := \lim_{n\to\infty} T_n x$ exists for all $x \in \mathbb{R}$, but the operator $T : \mathbb{R} \to \mathbb{R}$ defined in this way, i.e. $Tx = ax$, is not computable.*

Thus, for a computable version of the Banach-Steinhaus Theorem we need more information on the sequence $(T_i)_{i\in\mathbb{N}}$. It turns out that it suffice to know the uniform bound $\sup_{i\in\mathbb{N}} ||T_i||$ and the modulus of convergence of the sequences $(T_i x)_{i\in\mathbb{N}}$. Because of linearity, it even suffices to know these moduli on some fundamental sequence of unit vectors.

Theorem 12.3 (Computable Banach-Steinhaus Theorem) *Let X be a computable Banach space with some computable sequence $(e_j)_{j\in\mathbb{N}}$ of unit vectors whose linear span is dense in X and let Y be a computable normed space. Then the function*

$$L :\subseteq \mathcal{C}(X,Y)^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \to \mathcal{C}(X,Y), ((T_i)_{i\in\mathbb{N}}, m, s) \mapsto \left(x \mapsto \lim_{n\to\infty} T_n x\right),$$

defined for all tuples $((T_i)_{i\in\mathbb{N}}, m, s)$ such that the operators $T_i : X \to Y$ are linear bounded and pointwise convergent as $i \to \infty$, $\sup_{i\in\mathbb{N}} ||T_i|| \leq s$ and $||T_{m\langle i,j\rangle} e_j - \lim\limits_{n\to\infty} T_n e_j|| \leq 2^{-i}$ for all $i, j \in \mathbb{N}$, is computable.

Proof. Given a sequence $(T_i)_{i\in\mathbb{N}}$ of linear bounded and pointwise convergent operators $T_i : X \to Y$ together with a bound $s \geq \sup_{i\in\mathbb{N}} ||T_i||$ and a modulus of convergence $m : \mathbb{N} \to \mathbb{N}$ for $(e_j)_{j\in\mathbb{N}}$ such that $||T_{m\langle i,j\rangle} e_j - \lim_{n\to\infty} T_n e_j|| \leq 2^{-i}$, the classical Banach-Steinhaus Theorem guarantees that $T := L((T_i)_{i\in\mathbb{N}}, m, s) \in \mathcal{C}(X,Y)$ actually is defined and $||T|| \leq s$. W.l.o.g. we can assume that m is monotonically increasing. Given some $x \in X$ and some $k \in \mathbb{N}$ we can effectively find some finite linear combination $x' := \sum_{j=0}^{l} a_j e_j$ with $a_j \in Q_{\mathbb{F}}$ such that $||x - x'|| < \frac{1}{s} 2^{-k-2}$. Let $a := \max\{|a_j| : j = 0, ..., l\}$ and $k' \in \mathbb{N}$ such that $(l+1) \cdot a \cdot 2^{-k'} < 2^{-k-2}$, and let $M := \max\{m\langle k', j\rangle : j = 0, ..., l\}$ and $y := T_M x'$. Then we obtain

$$\begin{aligned} ||Tx - y|| &\leq ||Tx - Tx'|| + ||Tx' - y|| \\ &\leq s\frac{1}{s}2^{-k-2} + \left|\left|\sum_{j=0}^{l} a_j (Te_j - T_M e_j)\right|\right| \end{aligned}$$

$$\begin{aligned} &\leq 2^{-k-2} + (l+1) \cdot a \cdot 2^{-k'} \\ &< 2^{-k-1}. \end{aligned}$$

Thus, by producing some output $y' \in Y$ with $||y-y'|| < 2^{-k-1}$ one can actually compute T with precision 2^{-k}. Using evaluation and type conversion, one can show that a given tuple $((T_i)_{i\in\mathbb{N}}, m, s)$ can be effectively transformed into T. □

A combination of the computable Banach-Steinhaus Theorem 12.3 with the computable Uniform Boundedness Theorem 9.13 leads to the following corollary.

Corollary 12.4 *Let X be a computable Banach space with some computable sequence $(e_j)_{j\in\mathbb{N}}$ of unit vectors whose linear span is dense in X and let Y be a computable normed space. Then the function*

$$L' :\subseteq \mathcal{C}(X, \mathcal{B}(\mathbb{N}, Y)) \times \mathbb{N}^{\mathbb{N}} \to \mathcal{C}(X, Y), (T, m) \mapsto \left(x \mapsto \lim_{n\to\infty} T_n x\right),$$

defined for all linear bounded operators $T : X \to \mathcal{B}(\mathbb{N}, Y)$ such that $(T_i x)_{i\in\mathbb{N}}$ converges for each $x \in X$ as $i \to \infty$ and $||T_{m\langle i,j\rangle} e_j - \lim_{n\to\infty} T_n e_j|| \leq 2^{-i}$ for all $i, j \in \mathbb{N}$, is computable.

Since for a single sequence $(T_i)_{i\in\mathbb{N}}$ an upper bound s on the uniform bound is always available, we obtain the following less uniform version of the computable Banach-Steinhaus Theorem 12.3.

Corollary 12.5 *Let X be a computable Banach space with some computable sequence $(e_j)_{j\in\mathbb{N}}$ of unit vectors whose linear span is dense in X and let Y be a computable normed space. If $(T_i)_{i\in\mathbb{N}}$ is a computable sequence of linear and computable operators $T_i : X \to Y$ which converges pointwise and if $m : \mathbb{N} \to \mathbb{N}$ is a computable function such that $||T_{m\langle i,j\rangle} e_j - \lim_{n\to\infty} T_n e_j|| \leq 2^{-i}$ for all $i, j \in \mathbb{N}$, then by*

$$Tx := \lim_{n\to\infty} T_n x$$

a linear and computable operator $T : X \to Y$ is defined.

13 Landau's Theorem and Matrix Operators

In this section we will study functionals $T : \ell_p \to \mathbb{F}$ and matrix operators $T : \ell_p \to \ell_q$. We start with an investigation of Landau's Theorem.

Theorem 13.1 (Landau's Theorem) *Let $p, q > 1$ be real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ or let $p = 1$ and $q = \infty$ and let $a = (a_k)_{k\in\mathbb{N}}$ be some sequence in $\mathbb{F}$. Then $\sum_{k=0}^{\infty} a_k x_k$ converges for each $(x_k)_{k\in\mathbb{N}} \in \ell_p$, if and only if $a \in \ell_q$.*

The theorem is typically proved using Hölder's inequality and the Banach-Steinhaus Theorem (cf. Section 46.1 in [Heu86]). Given a sequence $a = (a_k)_{k\in\mathbb{N}} \in \ell_q$, let us denote by $\lambda_a \in (\ell_p)'$ the functional

$$\lambda_a : \ell_p \to \mathbb{F}, (x_k)_{k\in\mathbb{N}} \mapsto \sum_{k=0}^{\infty} a_k x_k.$$

Landau's Theorem leads to the question whether the mapping

$$\lambda : \ell_q \to \mathcal{C}(\ell_p, \mathbb{F}), a \mapsto \lambda_a$$

as well as its partial inverse λ^{-1} are computable? It turns out that at least the inverse λ^{-1} is not computable in general as the following simple example shows.

Example 13.2 *Let $p, q > 1$ be computable real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $a = (a_k)_{k\in\mathbb{N}}$ be a computable sequence of real numbers such that $||a||_q$ exists but is not computable. Then $\lambda_a : \ell_p \to \mathbb{R}$ is well-defined by Landau's Theorem and it is a computable linear operator by Theorem 8.7, since it is bounded by Hölder's inequality and $(\lambda_a e_k)_{k\in\mathbb{N}} = (a_k)_{k\in\mathbb{N}}$ is a computable sequence.*

Since one can show $||\lambda_a|| = ||a||_q$ (cf. [Heu86]), this example also shows that even simple computable functionals of Landau's type do not have a computable norm in general. Actually, one can use the idea of the previous example to prove that λ^{-1} is not continuous (see Theorem 13.5 below). Therefore the following computable version of Landau's Theorem is in a certain sense the best which one could expect.

Theorem 13.3 *Let $p, q > 1$ be computable real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ or let $p = 1$ and $q = \infty$. The mapping*

$$L :\subseteq \mathbb{F}^{\mathbb{N}} \times \mathbb{R} \to \mathcal{C}(\ell_p, \mathbb{F}), (a, s) \mapsto \lambda_a$$

with $\mathrm{dom}(L) := \{(a, s) \in \ell_q \times \mathbb{R} : ||a||_q \le s\}$ *is computable and it admits a multi-valued computable right-inverse.*

Proof. Given a sequence $a = (a_k)_{k\in\mathbb{N}} \in \mathbb{F}^{\mathbb{N}}$ and $s > 0$ such that $a \in \ell_q$ and $||\lambda_a|| = ||a||_q \le s$, and given some $x = (x_k)_{k\in\mathbb{N}} \in \ell_p$ and some precision $m \in \mathbb{N}$ we can effectively find some $n \in \mathbb{N}$ and numbers $q_0, ..., q_n \in Q_{\mathbb{F}}$ such that $||\sum_{i=0}^{n} q_i e_i - x||_p < \frac{1}{s} 2^{-m}$. It follows

$$\left|\lambda_a\left(\sum_{i=0}^{n} q_i e_i\right) - \lambda_a(x)\right| \le ||\lambda_a|| \cdot \left|\left|\sum_{i=0}^{n} q_i e_i - x\right|\right|_p < s \cdot \frac{1}{s} 2^{-m} = 2^{-m}$$

By linearity of λ_a we obtain $\lambda_a(\sum_{i=0}^n q_i e_i) = \sum_{i=0}^n q_i \lambda_a(e_i) = \sum_{i=0}^n q_i a_i$ and thus we can evaluate λ_a effectively up to any given precision m. Using type conversion this proves that L is computable.

Now let us assume that $\lambda_a \in \mathcal{C}(\ell_p, \mathbb{F})$ is given. By evaluation it follows that we can effectively compute $a = (a_k)_{k\in\mathbb{N}} = (\lambda_a(e_k))_{k\in\mathbb{N}}$. Moreover, Theorem 9.10 guarantees that we can effectively find some $s \in \mathbb{R}$ with $||a||_q = ||\lambda_a|| \leq s$. This shows that we can compute some multi-valued right inverse of L. □

Since the norm $||\ ||_q : \ell_q \to \mathbb{R}$ and the canonical injection $\ell_q \hookrightarrow \mathbb{F}^{\mathbb{N}}$ are computable, we obtain the following corollary, if we consider λ as mapping $\lambda : \ell_q \to (\ell_p)'$. This means that we replace $\mathcal{C}(\ell_p, \mathbb{F})$ by $(\ell_p)' = \mathcal{B}(\ell_p, \mathbb{F})$ in the image, together with the corresponding representations.

Corollary 13.4 (Computable Landau Theorem) *Let $p, q > 1$ be computable real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ or let $p = 1$ and $q = \infty$. Then the mapping*

$$\lambda : \ell_q \to (\ell_p)', a \mapsto \lambda_a$$

is a computable isometric isomorphism, especially λ and λ^{-1} are computable.

As another consequence of Theorem 13.3 we obtain the result that the operator norm is a discontinuous mapping even for functionals $f : \ell_p \to \mathbb{R}$.

Theorem 13.5 *Let $p \geq 1$ be a computable real number. The operator norm*

$$||\ || : \mathcal{C}(\ell_p, \mathbb{F}) \to \mathbb{R}, f \mapsto ||f||$$

is not continuous.

Proof. Let us assume that the mapping $||\ || :\subseteq \mathcal{C}(\ell_p, \mathbb{F}) \to \mathbb{R}$, defined for linear bounded $T : \ell_p \to \mathbb{F}$ would be continuous. Now let us consider the mapping $L :\subseteq \mathbb{F}^{\mathbb{N}} \times \mathbb{R} \to \mathcal{C}(\ell_p, \mathbb{F})$ from Theorem 13.3 and let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then $||L(a, s)|| = ||\lambda_a|| = ||a||_q$ and $||\ || \circ L :\subseteq \mathbb{F}^{\mathbb{N}} \times \mathbb{R} \to \mathcal{C}(\ell_p, \mathbb{F})$ is continuous. Especially, it follows that $N :\subseteq \mathbb{F}^{\mathbb{N}} \to \mathbb{R}, a \mapsto ||a||_q$ is continuous, say for all a with $||a||_q \leq 1$. But this is obviously not the case, since the product topology on $\{a \in \mathbb{F}^{\mathbb{N}} : ||a||_q \leq 1\}$ is strictly smaller than the corresponding subtopology of the ℓ_q topology. □

Now we continue to study the slightly more complicated case of matrix operators $T : \ell_p \to \ell_p$. We start with a definition.

Definition 13.6 (Matrix Operator) Let $p, q \geq 1$ be real numbers. A mapping $T : \ell_p \to \ell_q$ is called *matrix operator*, if there exists an infinite matrix $A = (a_{ik}) \in \mathbb{F}^{\mathbb{N}\times\mathbb{N}}$ such that for all $x = (x_k)_{k\in\mathbb{N}} \in \ell_p$

(1) $y_i := \sum\limits_{k=0}^{\infty} a_{ik}x_k$ exists for all $i \in \mathbb{N}$,

(2) $y := (y_i)_{i\in\mathbb{N}} \in \ell_q$ and

(3) $Tx = y$.

If an infinite matrix $A = (a_{ik}) \in \mathbb{F}^{\mathbb{N}\times\mathbb{N}}$ fulfills conditions (1) and (2), then we will denote the corresponding matrix operator $T : \ell_p \to \ell_q$, defined by condition (3), by $\hat{A} := T$. We mention that from the computational point of view we do not have to distinguish $\mathbb{F}^{\mathbb{N}\times\mathbb{N}}$ from $(\mathbb{F}^{\mathbb{N}})^{\mathbb{N}}$ and we will also use the notation $\hat{A}$ if $A \in (\mathbb{F}^{\mathbb{N}})^{\mathbb{N}}$. Using the Closed Graph Theorem one can prove that each matrix operator $T : \ell_p \to \ell_q$ is bounded [Heu86]. Now the question appears under which conditions on A computability of $\hat{A}$ is guaranteed? The following example shows that there exists a computable matrix $A \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ such that the corresponding matrix operator $\hat{A} : \ell_p \to \ell_q$ is well-defined but not computable. It is simply the "vertical" counterpart of the "horizontal" Example 13.2.

Example 13.7 *Let $p, q \geq 1$ be computable real numbers. Let $a = (a_i)_{i\in\mathbb{N}}$ be a computable sequence of real numbers such that $s := ||a||_q$ exists, but is not computable. Then the matrix $A = (a_{ik}) \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ with $a_{i0} := a_i$ for all $i \in \mathbb{N}$ and $a_{ik+1} := 0$ for all $i, k \in \mathbb{N}$ is a computable matrix but the matrix operator $\hat{A} : \ell_p \to \ell_q$ is not computable since $\hat{A}e_0 = a$ is not computable in ℓ_q.*

Using the fact that $\hat{A}e_k = (a_{ik})_{i\in\mathbb{N}}$ is the k–th column vector whenever $A = (a_{ik})$ is a matrix which defines some matrix operator $\hat{A}$, it is straightforward to characterize those matrices which induce computable matrix operators by Theorem 8.7.

Corollary 13.8 *Let $p, q \geq 1$ be computable real numbers and let the matrix $A = (a_{ik}) \in \mathbb{F}^{\mathbb{N}\times\mathbb{N}}$ be such that $\hat{A} : \ell_p \to \ell_q$ is well-defined. Then $\hat{A}$ is computable, if and only if $((a_{ik})_{i\in\mathbb{N}})_{k\in\mathbb{N}}$ is a computable sequence in ℓ_q.*

However, not any matrix whose column vectors form a computable sequence in ℓ_q define a matrix operator $\hat{A}$. By Landau's Theorem we know that in case $p > 1$ at least the row vectors have to be in $\ell_{\frac{p}{p-1}}$. On the other hand, these both conditions are not sufficient, since the unit matrix I consists of column and row vectors which form a computable sequence in ℓ_p for any p, but the corresponding matrix operator $\hat{I} = \mathrm{id} : \ell_p \to \ell_q$ is only well-defined (and computable) for $p \leq q$. Nevertheless we can formulate a theorem which generalizes the computable version of Landau's Theorem 13.3 for matrix operators. The proof is essentially the same.

Theorem 13.9 (Computable matrix operators) *Let $p > 1, q \geq 1$ be computable real numbers. The mapping*

$$M :\subseteq (\ell_q)^{\mathbb{N}} \times \mathbb{R} \to \mathcal{C}(\ell_p, \ell_q), (A, s) \mapsto \hat{A}$$

with $\mathrm{dom}(M) := \{(A, s) \in (\ell_q)^{\mathbb{N}} \times \mathbb{R} : ||\hat{A}|| \leq s\}$ *is computable and admits a multi-valued computable right-inverse.*

Proof. Given a matrix $A = (a_k)_{k\in\mathbb{N}} \in (\ell_q)^{\mathbb{N}}$ and $s > 0$ such that $||\hat{A}|| \leq s$, and given some $x = (x_k)_{k\in\mathbb{N}} \in \ell_p$ and some precision $m \in \mathbb{N}$ we can effectively find some $n \in \mathbb{N}$ and numbers $q_0, ..., q_n \in Q_{\mathbb{F}}$ such that $||\sum_{i=0}^n q_i e_i - x||_p < \frac{1}{s}2^{-m}$. It follows

$$\left|\left|\hat{A}\left(\sum_{i=0}^n q_i e_i\right) - \hat{A}(x)\right|\right|_q \leq ||\hat{A}|| \cdot \left|\left|\sum_{i=0}^n q_i e_i - x\right|\right|_p < s \cdot \frac{1}{s}2^{-m} = 2^{-m}$$

By linearity of $\hat{A}$ we obtain $\hat{A}(\sum_{i=0}^n q_i e_i) = \sum_{i=0}^n q_i \hat{A}(e_i) = \sum_{i=0}^n q_i a_i$ and thus we can evaluate $\hat{A}$ effectively up to any given precision m. Using type conversion one can prove that M is computable.

Now let us assume that $\hat{A} \in \mathcal{C}(\ell_p, \ell_q)$ is given. By evaluation it follows that we can effectively compute $A = (a_k)_{k\in\mathbb{N}} = (\hat{A}(e_k))_{k\in\mathbb{N}} \in (\ell_q)^{\mathbb{N}}$. Moreover, theorem 9.10 guarantees that we can effectively find some $s \in \mathbb{R}$ with $||\hat{A}|| \leq s$. This shows that we can compute some multi-valued right inverse of M. □

It is worth mentioning that we have expressed $\mathrm{dom}(M)$ using the operator norm $||\hat{A}||$ while in the corresponding Theorem 13.3 we have used the identity $||\lambda_a|| = ||a||_q$ to replace the value $||\lambda_a||$ by $||a||_q$, thus by a condition which directly refers to the sequence a. Unfortunately, no such handy sufficient and necessary condition on matrices A is known in case of matrix operators, not even in the Hilbert space case $p = q = 2$ (cf. Problem 44 in [Hal82]). However, there is a sufficient criterion: we close this section with showing that a certain Banach space of matrices can be computably embedded into the space of matrix operators. Therefore let $\ell_{p,q} := \{(a_{ik})_{(i,k)\in\mathbb{N}^2} \in \mathbb{F}^{\mathbb{N}\times\mathbb{N}} : ||A||_{p,q} < \infty\}$ with

$$||A||_{p,q} := \left|\left|\Big(||(a_{ik})_{k\in\mathbb{N}}||_p\Big)_{i\in\mathbb{N}}\right|\right|_q = \left(\sum_{i=0}^{\infty}\left(\sum_{k=0}^{\infty}|a_{ik}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}.$$

It is easy to see that for computable real numbers p, q the space $(\ell_{p,q}, ||\ ||_{p,q}, e)$ is a computable Banach space, where (e_{ik}) is the matrix with 1 in position (i, k) and 0 elsewhere. In the case $p = q = 2$ the space $\ell_{2,2}$ is isometric isomorphic to the space $\mathcal{H}(\ell_2)$ of Hilbert-Schmidt operators $T : \ell_2 \to \ell_2$ with the Hilbert-Schmidt norm (cf. [Heu86]). First we show that for suitable values of p, q each matrix $A \in \ell_{p,q}$ induces a matrix operator $\hat{A}$.

Lemma 13.10 *Let $p > 1, q \geq 1$ be real numbers. If $A \in \ell_{\frac{p}{p-1},q}$, then the matrix operator $\hat{A} : \ell_p \to \ell_q$ is well-defined and $||\hat{A}|| \leq ||A||_{\frac{p}{p-1},q}$.*

Proof. Let $A = (a_{ik}) \in \ell_{\frac{p}{p-1},q}$ and $x = (x_k)_{k\in\mathbb{N}} \in \ell_p$. Since

$$||(a_{ik})_{k\in\mathbb{N}}||_{\frac{p}{p-1}} \leq \left|\left|\left(||(a_{ik})_{k\in\mathbb{N}}||_{\frac{p}{p-1}}\right)_{i\in\mathbb{N}}\right|\right|_q = ||A||_{\frac{p}{p-1},q} < \infty$$

we obtain $(a_{ik})_{k\in\mathbb{N}} \in \ell_{\frac{p}{p-1}}$ for all $i \in \mathbb{N}$. By Hölders inequality this implies that $y_i := \sum_{k=0}^{\infty} a_{ik}x_k$ exists for all $i \in \mathbb{N}$ and with $y := (y_i)_{i\in\mathbb{N}}$ we obtain

$$\begin{aligned}
||y||_q &= \left(\sum_{i=0}^{\infty} |y_i|^q\right)^{\frac{1}{q}} \\
&= \left(\sum_{i=0}^{\infty} \left|\sum_{k=0}^{\infty} a_{ik}x_k\right|^q\right)^{\frac{1}{q}} \\
&\leq \left(\sum_{i=0}^{\infty} \left(\sum_{k=0}^{\infty} |a_{ik}x_k|\right)^q\right)^{\frac{1}{q}} \\
&\leq \left(\sum_{i=0}^{\infty} \left(||(a_{ik})_{k\in\mathbb{N}}||_{\frac{p}{p-1}} \cdot ||x||_p\right)^q\right)^{\frac{1}{q}} \\
&= ||x||_p \cdot \left(\sum_{i=0}^{\infty} \left(\sum_{k=0}^{\infty} |a_{ik}|^{\frac{p}{p-1}}\right)^{\frac{(p-1)q}{p}}\right)^{\frac{1}{q}} \\
&= ||x||_p \cdot ||A||_{\frac{p}{p-1},q}.
\end{aligned}$$

Hence $y \in \ell_q$ and $\hat{A} : \ell_p \to \ell_q$ is well-defined. Moreover, $||\hat{A}|| \leq ||A||_{\frac{p}{p-1},q}$. □

Using this lemma we obtain the following corollary of Theorem 13.9.

Corollary 13.11 *Let $p > 1, q \geq 1$ be computable real numbers. The mapping*

$$H : \ell_{\frac{p}{p-1},q} \to \mathcal{B}(\ell_p, \ell_q), A \mapsto \hat{A}$$

is computable.

This corollary provides a handy sufficient and uniform criterion which guarantees that the matrix operator map $A \mapsto \hat{A}$ becomes computable.

14 The Finite-Dimensional Case

In the previous sections we have investigated operators $T : X \to Y$ mainly in the general case of separable possibly infinite-dimensional Banach spaces X, Y. In this section we want to investigate whether we can improve our results in case of finite-dimensional spaces X, Y. First of all we will prove that we can restrict ourselves to the spaces $X = \mathbb{F}^n$ and $Y = \mathbb{F}^m$ since any n-dimensional computable normed space X is computably isomorphic to $\mathbb{F}^n$. As a preparation we prove that any finite-dimensional computable Banach space has a computable algebraic basis. The proof is based on the Lemma of Riesz [GP65].

Proposition 14.1 (Computable basis) *Let X be an n–dimensional computable Banach space. Then there exist computable vectors $x_1, ..., x_n \in X$ such that $(x_1, ..., x_n)$ is a vector space basis of X.*

Proof. Consider the computable Banach space $(X, ||\ ||, e)$. Let $i_1, ..., i_k \in \mathbb{N}$ be such that $(e_{i_1}, ..., e_{i_k})$ is a linear independent tuple of vectors of maximal cardinality k. If $k = n$ then $(e_{i_1}, ..., e_{i_k})$ is a basis of X and we are finished. Thus, let us assume $k < n$ and let L be the linear span of $\{e_{i_1}, ..., e_{i_k}\}$. Since k is maximal with the property above, it follows that $e_i \in L$ for all $i \in \mathbb{N}$. Now on the one hand the linear span of range(e) is dense in X and thus $\overline{L} = X$ and on the other hand $k < n$ implies that L is a proper linear subspace of X and hence by the Lemma of Riesz there exists some point $x \in X$ such that $d_L(x) = \inf_{y \in L} ||x - y|| = 1$ and hence $x \notin \overline{L}$. Contradiction! □

If $(x_1, ..., x_n)$ is a vector space basis with computable vectors $x_1, ..., x_n \in X$, then we will call it a *computable vector space basis*. We recall that any finite-dimensional normed space is separable and complete, i.e. a separable Banach space. We will use the basis $(e_1, ..., e_n)$ of $\mathbb{F}^n$, as defined in Proposition 3.8(1).

Theorem 14.2 (Finite-dimensional computable Banach spaces) *Let X be an n–dimensional computable Banach space with a computable vector space basis $(x_1, ..., x_n)$. Then the unique linear function $f : X \to \mathbb{F}^n$ with $f(x_i) := e_i$ is a computable isomorphism, i.e. f as well as f^{-1} are computable.*

Proof. Since in finite-dimensional Banach spaces any linear function is continuous, $||f||$ exists. Since f is bijective, we obtain $||f|| > 0$. Let $x \in X$ and a precision $k \in \mathbb{N}$ be given. Since X is a computable normed space, we can effectively find coefficients $a_1, ..., a_n \in \mathbb{F}$ such that $|| \sum_{i=1}^n a_i x_i - x|| < \frac{1}{||f||} 2^{-k}$. By linearity of f it follows

$$\left| f(x) - \sum_{i=1}^n a_i e_i \right| \leq ||f|| \cdot \left|\left| x - \sum_{i=1}^n a_i x_i \right|\right| < 2^{-k}.$$

Since addition and scalar multiplication is computable in $\mathbb{F}^n$, we can compute $\sum_{i=1}^{n} a_i e_i$ and this approximates $f(x)$ with precision 2^{-k}. Thus, f is computable. Computability of the inverse f^{-1} can be proved analogously. □

This theorem allows to restrict our investigation of finite-dimensional spaces to the cases $X = \mathbb{F}^n$ and $Y = \mathbb{F}^m$. Elements of computable linear algebra for the finite-dimensional case can be found in [BZ00, ZB00, ZB01].

14.1 The Finite-Dimensional Open Mapping Theorem

In this subsection we will show that we can improve our results on the uniform computable version of the Open Mapping Theorem in case of finite-dimensional target spaces. We recall that $\mathcal{O}(T) : \mathcal{O}(X) \to \mathcal{O}(\mathbb{F}^m), U \mapsto T(U)$ is the operation associated with any open map $T : X \to \mathbb{F}^m$.

Theorem 14.3 (Finite-dimensional Open Mapping Theorem) *Let X be a computable normed space and $m \geq 1$ be a natural number. The map*

$$\Omega : T \mapsto \mathcal{O}(T),$$

which is defined for all linear bounded and surjective operators $T : X \to \mathbb{F}^m$, is $([\delta_X \to \delta_{\mathbb{F}}^m], [\delta_{\mathcal{O}(X)} \to \delta_{\mathcal{O}(\mathbb{F}^m)}])$–computable.

Proof. We consider the real-valued case $\mathbb{F} = \mathbb{R}$. Given a linear bounded and surjective operator $T : X \to \mathbb{R}^m$, it suffices to find some rational $r > 0$ such that $B(0, r) \subseteq TB(0, 1)$. We recall that by the classical Open Mapping Theorem T is open and thus the desired result follows by Theorem 5.5. It is easy to see that there exists a computable sequence $f : \mathbb{N} \to X$ such that range(f) is dense in $B(0, 1)$. Thus, by continuity of T the sequence $T \circ f$ is dense in $TB(0, 1)$. We define a partial relation $<$ on $\mathbb{R}^m$ by

$$(x_1, ..., x_m) < (y_1, ..., y_m) :\iff (\forall i = 1, ..., m)\ 1 < \frac{y_i}{x_i}.$$

Since $\mathbb{R}^m$ is endowed with the maximum metric, the ball $B(0, r)$ is an open cube with vertices $v_1, ..., v_{2^m}$ and all components of the vertices v_i have absolute value r. Moreover, $B(0, r)$ is the interior of the convex hull conv$\{v_1, ..., v_{2^m}\}$ of $v_1, ..., v_{2^m}$. If there are vectors $x_1, ..., x_{2^m} \in TB(0, 1)$ such that $x_i > v_i$ for all $i = 1, ..., 2^m$, then we obtain

$$B(0, r) \subseteq \text{conv}\{v_1, ..., v_{2^m}\} \subseteq \text{conv}\{x_1, ..., x_{2^m}\} \subseteq TB(0, 1),$$

where the last inclusion holds since T is linear and thus $TB(0, 1)$ is convex (as image of the convex ball $B(0, 1)$). On the other hand, since T is open and $T(0) = 0$, there is always some $s > 0$ such that $B(0, s) \subseteq TB(0, 1)$ and thus there are points $x_1, ..., x_{2^m} \in \text{range}(Tf)$ which approximate the vertices of $B(0, s)$ with distance less than $\frac{s}{2}$. Thus, we obtain $x_i < v_i$ for the vertices $x_1, ..., x_{2^m}$ of the ball $B(0, r)$ with $r := \frac{s}{2}$. Altogether, this proves that it suffices to find vectors $x_1, ..., x_{2^m} \in \text{range}(Tf)$ such that $x_i < v_i$. Given $T \in \mathcal{C}(X, \mathbb{R}^m)$, we can effectively find Tf by evaluation and type conversion. Since $<$ is obviously an r.e. open subset of $\mathbb{R}^m \times \mathbb{R}^m$ and the mapping $V :\subseteq \mathbb{Q} \to (\mathbb{R}^m)^{2^m}, r \mapsto (v_1, ..., v_{2^m})$, which maps each $r > 0$ to the vertices of the ball $B(0, r)$ is computable, we can effectively find some $r > 0$ and $n_1, ..., n_{2^m} \in \mathbb{N}$ such that $x_i > v_i$ holds for the points $x_i := Tf(n_i)$ and this r actually fulfills $B(0, r) \subseteq TB(0, 1)$. The complex case $\mathbb{F} = \mathbb{C}$ can be deduced from the real number case, since $\mathbb{C}$ can be identified with $\mathbb{R}^2$ for the purposes of this proof ($\mathbb{C}$ is endowed with the Euclidean metric and $\mathbb{R}^2$ with the maximum metric but since $B_{\mathbb{C}}(x, r) \subseteq B_{\mathbb{R}^2}(x, r)$ holds for the corresponding balls, no problem occurs). □

A similar result can be proved in case of analytic functions too (cf. the Effective Open Mapping Theorem 4.3 in [Her99]). Together with Theorem 6.4 and Theorem 14.2 we obtain the following computable version of Banach's Inverse Mapping Theorem using the composition $\omega^{-1} \circ \Omega$.

Corollary 14.4 (Finite-dimensional Inverse Mapping Theorem) *Let X, Y be finite-dimensional computable normed spaces. The mapping*

$$\iota :\subseteq \mathcal{C}(X, Y) \to \mathcal{C}(Y, X), T \mapsto T^{-1},$$

which is defined for all linear (and bounded) bijective operators $T : X \to Y$, is $([\delta_X \to \delta_Y], [\delta_Y \to \delta_X])$–computable.

It should be observed that it would be not strengthening to demand that the space X *or* the space Y is finite-dimensional since in this case a bijective linear operator $T : X \to Y$ can only exist if X and Y are of the same finite dimension n. Using the latter fact we could alternatively conclude this corollary from the special case $X = Y = \mathbb{F}^n$ by virtue of Theorem 14.2. Finally, the case $X = Y = \mathbb{F}^n$ can easily be proved directly using a matrix representation of T (cf. [ZB00, BZ00, ZB01]).

From the previous corollary we can especially deduce that our main source for counterexamples, the class of diagonal operators $T_a : \ell_p \to \ell_p$ in Lemma 5.6 and Proposition 5.7, cannot be replaced by some other class of operators $T_a : X \to Y$ with finite-dimensional X or Y (cf. Corollary 6.5).

14.2 The Finite-Dimensional Graph Theorem

In Theorem 8.3 we have proved that the mapping

$$\text{graph} : \mathcal{C}(X,Y) \to \mathcal{A}(X \times Y), T \mapsto \text{graph}(T)$$

is computable, while Theorem 8.5 shows that the inverse mapping graph^{-1} is not continuous in case $X = Y = \ell_p$. In this subsection we want to study how the situation changes in the finite-dimensional case. Our aim is to prove a theorem which shows that in case $Y = \mathbb{F}^n$ the inverse becomes computable too (which is a generalization of Exercise 6.1.8 in [Wei00]). Therefore, we will use the hyperspace $\mathcal{K}(X) := \{K \subseteq X : K \text{ non-empty and compact}\}$ of non-empty compact subsets of a metric space X, endowed with the *Hausdorff metric* $d_\mathcal{K} : \mathcal{K}(X) \times \mathcal{K}(X) \to \mathbb{R}$, defined by

$$d_\mathcal{K}(A,B) := \max\left\{\sup_{a\in A} d_B(a), \sup_{b\in B} d_A(b)\right\}.$$

If (X,d,α) is a computable metric space, then a standard numbering $\alpha_\mathcal{K}$ of the set $\mathcal{Q}$ of finite subsets of $\text{range}(\alpha)$ can be defined by $\alpha_\mathcal{K}\langle k, \langle n_0,...,n_k\rangle\rangle := \{\alpha(n_0),...,\alpha(n_k)\}$. It is easy to see that under these assumptions $(\mathcal{K}(X), d_\mathcal{K}, \alpha_\mathcal{K})$ is a computable metric space too [Bra99b]. In the following we assume that $\mathcal{K}(X)$ is endowed with the corresponding Cauchy representation $\delta_\mathcal{K}$. It is easy to prove that the canonical injection $\mathcal{K}(X) \hookrightarrow \mathcal{A}(X)$ is computable [Bra99b, BP00]. As a preparation of our main result we formulate a lemma. Here, ∂A denotes the *border* of a subset $A \subseteq X$ of a metric space X.

Lemma 14.5 *Let $n \geq 1$ a natural number. The function*

$$I :\subseteq \mathbb{F}^n \times \mathbb{R} \to \mathcal{K}(\mathbb{F}^n), (y,\varepsilon) \mapsto \partial B(y,\varepsilon),$$

defined for all $(y,\varepsilon) \in \mathbb{F}^n \times \mathbb{R}$ with $\varepsilon > 0$, is computable.

The proof is straightforward. Now we are prepared to prove the following theorem. We will call a metric space X *everywhere connected*, if all open balls $B(x,\varepsilon)$ are connected subsets of X.

Theorem 14.6 (Finite-dimensional Graph Theorem) *Let X be an everywhere connected computable metric space and let $n \geq 1$ be a natural number. The mapping*

$$\text{graph} : \mathcal{C}(X,\mathbb{F}^n) \to \mathcal{A}(X \times \mathbb{F}^n), T \mapsto \text{graph}(T)$$

as well as its inverse $\text{graph}^{-1} :\subseteq \mathcal{A}(X \times \mathbb{F}^n) \to \mathcal{C}(X,\mathbb{F}^n)$ *are computable.*

Proof. We consider the computable metric space (X, d, α). Computability of graph follows from Theorem 8.3. We investigate the mapping graph^{-1}. Thus, let $f : X \to \mathbb{F}^n$ be a continuous function. We can assume that $\text{graph}(f)$ is given as point $\text{graph}(f) \in \mathcal{A}(X \times \mathbb{F}^n)$. Furthermore, let $x \in X$ and a precision $k \in \mathbb{N}$ be given. We prove that we can effectively evaluate $f(x)$ up to precision 2^{-k} which implies by type conversion the desired result. Since $\text{graph}(f)$ is given as point of $\mathcal{A}(X \times \mathbb{F}^n)$, we can, on the one hand, find points $(x_i, y_i) \in \text{range}(\alpha) \times Q_{\mathbb{F}}^n$ and rational numbers $r_i > 0$ such that

$$\text{graph}(f)^c = (X \times \mathbb{F}^n) \setminus \text{graph}(f) = \bigcup_{i=0}^{\infty} B((x_i, y_i), r_i) = \bigcup_{i=0}^{\infty} (B(x_i, r_i) \times B(y_i, r_i))$$

and, on the other hand, we can enumerate all positive information, i.e. all "rational" balls $U = B((x', y), r) = B(x', r) \times B(y, r) \subseteq X \times \mathbb{F}^n$ such that $U \cap \text{graph}(f) \neq \emptyset$. Given a compact subset $K \in \mathcal{K}(\mathbb{F}^n)$ such that $K \subseteq \bigcup_{i=0}^{\infty} B(y_i, r_i)$, the computable Heine-Borel Theorem (cf. Theorems 4.6 and 4.10 in [BW99]) ensures that we can effectively find a natural number $m \in \mathbb{N}$ such that $K \subseteq \bigcup_{i=0}^{m} B(y_i, r_i)$. Now the evaluation of $f(x)$ up to precision 2^{-k} works as follows: we systematically search for some $(x', y) \in \text{range}(\alpha) \times Q_{\mathbb{F}}^n$, some rational numbers $r > 0$, $t \geq 1$ and natural numbers $n_0, ..., n_m \in \mathbb{N}$ such that

(1) $U := B(x', r) \times B(y, r)$,

(2) $U \cap \text{graph}(f) \neq \emptyset$,

(3) $x \in B(x', r)$,

(4) $d(x_{n_i}, x') + r < r_{n_i}$ for $i = 0, ..., m$,

(5) $\partial B(y, tr) \subseteq \bigcup_{i=0}^{m} B(y_{n_i}, r_{n_i})$,

(6) $tr < 2^{-k}$.

If such $x', y, r, t, n_0, ..., n_m$ exist, then we can effectively find such values by the previous considerations and the previous lemma. Now we claim that such values always exist and $||f(x) - y|| < 2^{-k}$ if (1) to (6) is fulfilled.

Thus, let us first assume that (1) to (6) holds. It is obvious that (4) implies $B(x', r) \subseteq \bigcap_{i=0}^{m} B(x_{n_i}, r_{n_i})$ and thus by (5) it follows

$$B(x', r) \times \partial B(y, tr) \subseteq \bigcup_{i=0}^{m} (B(x_{n_i}, r_{n_i}) \times B(y_{n_i}, r_{n_i})) \subseteq \text{graph}(f)^c.$$

Thus, $fB(x', r) \subseteq B(y, tr) \cup (\mathbb{F}^n \setminus \overline{B}(y, tr))$ and hence $fB(x', r)$ is covered by a disjoint union of open subsets. Since $B(x', r)$ is connected by presumption and

f is continuous, it follows that $fB(x',r)$ is connected too and thus $fB(x',r)$ has empty intersection with either $B(y,tr)$ or $\mathbb{F}^n \setminus \overline{B}(y,tr)$. By (1) and (2) there is some $x'' \in B(x',r)$ such that $f(x'') \in B(y,r) \subseteq B(y,tr)$ and hence $fB(x',r) \cap \overline{B}(y,tr)^c = \emptyset$. This implies $f(x) \in B(y,tr)$ by (3) and hence $||f(x) - y|| < tr < 2^{-k}$ by (6).

Now we still have to prove that there are always values $x', y, r, t, n_0, ..., n_m$ such that (1) to (6) are fulfilled. Therefore, let $0 < s < 2^{-k}$ be a rational number. Then $\partial B(f(x), s) \subseteq \bigcup\{B(y_i, r_i) : i \in \mathbb{N} \text{ and } x \in B(x_i, r_i)\}$ and by compactness there exist $n_0, ..., n_m \in \mathbb{N}$ such that $\partial B(f(x), s) \subseteq \bigcup_{i=0}^m B(y_{n_i}, r_{n_i})$ and $x \in \bigcap_{i=0}^m B(x_{n_i}, r_{n_i}) =: V$. Thus, V is an open neighbourhood of x and since range(α) is dense in X we can choose some $x' \in$ range(α) and some rational r with $0 < r < s$ such that $d(x, x') < r$ and $d(x_{n_i}, x') + r < r_{n_i}$ for all $i = 0, ..., m$, especially $x \in B(x', r) \subseteq V$. Hence there exists some $y \in Q_{\mathbb{F}}^n$ such that $d(f(x), y) < r$ and still $\partial B(y, s) \subseteq \bigcup_{i=0}^m B(y_{n_i}, r_{n_i})$. For this choice of $x', y, r, n_0, ..., n_m$, $t := \frac{s}{r}$ and $U := B((x', y), r)$ all conditions (1) to (6) are fulfilled: (1) is the definition of U. (2) and (3) holds since by definition $x \in B(x', r)$ and $f(x) \in B(y, r)$ and thus $(x, f(x)) \in$ graph$(f) \cap U$. (5) holds by choice of y and t and (4) by choice of x'. (6) holds since $tr = s < 2^{-k}$ by choice of s and definition of t. □

In case that all balls in X are relatively compact, e.g. if $X = \mathbb{F}^m$, we could simplify this proof. In this case the finite subcovering, i.e. the number m, could be determined uniformly for a whole neighbourhood $B(x, r)$ of x. But this simpler version would be to weak for our purposes since infinite-dimensional spaces X are not locally compact and hence their balls are not relatively compact in general. However, computable normed spaces X are everywhere connected since all balls are convex. As combination of the previous theorem with Theorem 8.3 and Lemma 8.4 we obtain the following corollary which shows that inversion is effective for finite-dimensional source spaces even in the non-linear case.

Corollary 14.7 *Let Y be an everywhere connected computable metric space and $n \geq 1$ be a natural number. The mapping*

$$\iota :\subseteq \mathcal{C}(\mathbb{F}^n, Y) \to \mathcal{C}(Y, \mathbb{F}^n), T \mapsto T^{-1},$$

defined for all homeomorphisms $T : \mathbb{F}^n \to Y$, is $([\delta_{\mathbb{F}}^n \to \delta_Y], [\delta_Y \to \delta_{\mathbb{F}}^n])$–computable.

14.3 Computing Bounds

In Theorem 9.7 we have seen that that mapping $T \mapsto ||T||$, defined for linear bounded operators $T : X \to Y$, is not continuous in general. It is easy to see that this situation changes if X is finite-dimensional. The following result improves Theorem 9.3 for the finite-dimensional case.

Theorem 14.8 (Computable bound) *Let Y be a computable normed space and let $n \geq 1$ be a natural number. The partial map*

$$|| \, || :\subseteq \mathcal{C}(\mathbb{F}^n, Y) \to \mathbb{R}, T \mapsto ||T||$$

is computable.

Proof. Given $T : \mathbb{F}^n \to Y$ we can effectively find the function $S := || \, || \circ T$, i.e. $S : \mathbb{F}^n \to \mathbb{R}, x \mapsto ||Tx||$ by evaluation and type conversion since $|| \, || : Y \to \mathbb{R}$ is computable. Now $\overline{B}(0,1)$ is a compact recursive subset of $\mathbb{F}^n$ by Lemma 9.2 and thus

$$||T|| = \sup_{x \in \overline{B}(0,1)} ||Tx|| = \max S(\overline{B}(0,1))$$

can be computed (e.g. by virtue of Corollary 6.2.5 of [Wei00]). □

Theorem 9.11 already proves that a corresponding improvement of the semi-computable Uniform Boundedness Theorem 9.8 is impossible. As a corollary of the previous theorem we immediately obtain that the bound of any computable bounded operator $T : \mathbb{F}^n \to Y$ is a computable real number.

Corollary 14.9 *Let Y be a computable normed space and let $n \geq 1$ be a natural number. If $T : \mathbb{F}^n \to Y$ is a computable bounded operator, then $||T||$ is a computable real number.*

While finite-dimensionality of the source space suffices to guarantee computability of the bound, Example 13.2 shows that finite-dimensionality of the target space does not help.

15 The Non-Separable Case

Our whole investigation of Banach space principles is focused on infinite-dimensional separable Banach spaces. Separability is necessary since computable Banach spaces are separable by definition. However, we have also successfully computed with some non-separable Banach spaces as ℓ_∞, $\mathcal{B}(\mathbb{N}, Y)$ and $\mathcal{B}(X, Y)$ (for infinite-dimensional X). Now two natural questions appear:

(1) Can we extend the notion of a computable Banach space to non-separable spaces in a reasonable way?

(2) Can we extend our results on Banach space principles to this definition?

In the following we will answer the first question in the affirmative, by presenting a definition of a general computable Banach space, which generalizes our definition for the separable case and which places the ad hoc treatment of ℓ_∞, $\mathcal{B}(\mathbb{N}, Y)$ and $\mathcal{B}(X, Y)$ in a uniform framework. On the other hand, this specific generalized definition leads to a negative answer concerning the second question at least if we want to have both: a natural duality result as expressed in the computable version of Landau's Theorem 13.4, as well as a computable version of Banach's Inverse Mapping Theorem as formulated in Corollary 6.3.

The possibility to handle non-separable normed spaces is mainly limited by the following fact (cf. Lemma 8.1.1 in [Wei00]).

Proposition 15.1 *Each represented metric space (X, δ) with a continuous representation δ is separable.*

This follows since separability of the Cantor space leads to separability of the represented space. As a consequence one obtains the following corollary.

Corollary 15.2 *Each represented normed space (X, δ) such that*

(1) the norm $|| \, || : X \to \mathbb{R}$ is $(\delta, \delta_\mathbb{R})$–continuous and

(2) the vector space subtraction $- : X \times X \to X$ is $([\delta, \delta], \delta)$–continuous,

is separable.

The proof follows from the observation that the two conditions imply that the representation is continuous (cf. Lemma 8.1.1 in [Wei00]). The following result shows that the problem is not just a topological problem since it does not disappear if we are only interested in computable points.

Proposition 15.3 *There exists no representation δ of ℓ_∞ such that*

(1) the δ–computable points $x \in \ell_\infty$ are exactly the $\delta_\mathbb{F}^\mathbb{N}$–computable points with computable norm $||x||_\infty$,

(2) the vector addition $+ : \ell_\infty \times \ell_\infty \to \ell_\infty$ maps δ–computable points to δ–computable points.

Proof. Let us assume that δ is a representation of ℓ_∞ such that (1) and (2) holds. Let $x = (x_0, x_1, ...) \in \ell_\infty$ be some $\delta_\mathbb{F}^\mathbb{N}$–computable point such that $x_0 \in \mathbb{F}$ is computable but $||x||_\infty$ is not computable and let $r > ||x||_\infty$ be some rational number. By (1) $y = (-2r, 0, 0, ...) \in \ell_\infty$ is δ–computable and

$z = (z_0, z_1, ...) := x - y$ is $\delta_{\mathbb{F}}^{\mathbb{N}}$–computable. Since $|x_0| \leq ||x||_\infty < r$ we obtain that $||z||_\infty = |z_0| = |x_0 + 2r|$ is computable and hence z is also δ–computable by (1). By (2) it follows that $x = z + y$ is δ–computable too which is a contradiction! □

Since the δ_{ℓ_∞}–computable points are exactly the $\delta_{\mathbb{F}}^{\mathbb{N}}$–computable points with computable norm, we obtain the following corollary.

Corollary 15.4 *The δ_{ℓ_∞}–computable points are not closed under addition.*

Regarding these negative results, we have to decide whether we want to keep computability of the norm or of the vector space operations in case of non-separable normed spaces. To be as general as necessary, we take the following definition for a general computable normed space.

Definition 15.5 (General computable normed space) A tuple $(X, ||\ ||, \delta)$ is called a *general computable normed space*, if (X, δ) is a computable vector space and $\mathrm{Lim} :\subseteq X^{\mathbb{N}} \to X$ is computable.

We recall that

$$\mathrm{dom}(\mathrm{Lim}) = \{(x_n)_{n\in\mathbb{N}} : (\forall i > j)\, ||x_i - x_j|| \leq 2^{-j}\}.$$

If in the situation of the definition $(X, ||\ ||)$ is even a Banach space, then we call $(X, ||\ ||, \delta)$ a *general computable Banach space*. Each general computable normed space gives rise to at least two canonical representations which yield certain computability properties of the norm.

Definition 15.6 *Let $(X, ||\ ||, \delta)$ be a general computable normed space. We define two representations $\delta^=, \delta^{\geqslant}$ of X by*

(1) $\delta^=\langle p, q\rangle = x :\iff \delta(p) = x$ *and* $\delta_{\mathbb{R}}(q) = ||x||$,

(2) $\delta^{\geqslant}\langle p, q\rangle = x :\iff \delta(p) = x$ *and* $\delta_{\mathbb{R}}(q) \geq ||x||$.

The next proposition shows that $\delta^=$ just contains sufficient information on the represented points to compute their norm.

Proposition 15.7 (Effective Normability) *Let $(X, ||\ ||, \delta)$ be a general computable normed space. Then $\delta^= \leq \delta^{\geqslant} \leq \delta$ and $\delta^=$ is maximal among all representations below δ which make the norm $||\ || : X \to \mathbb{R}$ computable.*

Proof. Obviously, $\delta^{=} \leq \delta^{\geqslant} \leq \delta$ holds; the first translation is realized by the identity, the second by the projection on the first component. The norm becomes computable w.r.t. $\delta^{=}$ and is realized by the projection on the second component. Now let $\delta' \leq \delta$ be an arbitrary representation which makes the norm computable. If $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ realizes the translation of δ' into δ and $G :\subseteq \Sigma^\omega \to \Sigma^\omega$ realizes the norm w.r.t. δ', then $H :\subseteq \Sigma^\omega \to \Sigma^\omega$, defined by $H(p) := \langle F(p), G(p) \rangle$, translates δ' to $\delta^{=}$. Hence $\delta^{=}$ is maximal below δ among all representations which make the norm computable. □

The previous proposition highly relies on the fact that the norm is a unary function. A unary function can always be effectivized by a construction analogously to the definition of $\delta^{=}$, but the same construction does not work for functions which depend on two inputs. This can be deduced from Proposition 15.1 using some non-separable metric. The following example additionally admits a simple limit operation.

Example 15.8 *Let* $(\mathbb{R}, \delta)$ *be a represented space and let* $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be the discrete metric on* $\mathbb{R}$*, i.e.* $d(x,x) = 0$ *and* $x \neq y \Longrightarrow d(x,y) = 1$ *for all* $x, y \in \mathbb{R}$*. Then* $\mathrm{Lim} :\subseteq \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ *is simply the operation* $\mathrm{Lim}(x_n)_{n\in\mathbb{N}} = x_1$ *and thus computable w.r.t. any representation and hence w.r.t.* δ*. But there is no representation* δ' *of* $\mathbb{R}$ *such that* d *becomes computable w.r.t.* δ'*, since otherwise* $\{(x,y) \in \mathbb{R} \times \mathbb{R} : x = y\}$ *would be a* δ'*–r.e. set, which is impossible (cf. Theorem 4.1.16 in [Wei00]).*

Alternatively, one could argue with Proposition 15.1 and the fact that the discrete space $(\mathbb{R}, d)$ is non-separable. The next observation shows that each computable normed space is especially a general computable normed space. Thus, the notion of a general computable normed space actually generalizes the ordinary notion.

Proposition 15.9 (Computable normed spaces) *If* $(X, ||\ ||, e)$ *is a computable normed space with Cauchy representation* δ*, then* $(X, ||\ ||, \delta)$ *is a general computable normed space and* $\delta^{=} \equiv \delta^{\geqslant} \equiv \delta$*.*

By definition of computable normed spaces it follows that the Cauchy representation δ yields a computable vector space (X, δ). By Proposition 3.2 the limit operation and the metric, induced by the norm, are computable w.r.t. the Cauchy representation δ. The latter implies that the norm itself is computable since $0 \in X$ is a computable point. By Proposition 15.7 this implies $\delta \leq \delta^{=}$ and thus $\delta \equiv \delta^{=}$. The next proposition shows that a general computable normed space keeps as much of the computability properties of a computable normed space as possible (in view of Corollary 15.2).

Proposition 15.10 *Let $(X, || \ ||, \delta)$ be a general computable normed space. Then*

(1) $|| \ || : X \to \mathbb{R}$ *is* $(\delta^=, \delta_\mathbb{R})$*–computable,*

(2) $\mathrm{Lim} :\subseteq X^\mathbb{N} \to X$ *is* $(\delta^{=\mathbb{N}}, \delta^=)$*–computable,*

(3) $+ : X \times X \to X, (x, y) \mapsto x + y$ *is* $([\delta^\geqslant, \delta^\geqslant], \delta^\geqslant)$*–computable,*

(4) $\cdot : \mathbb{F} \times X \to X, (a, x) \mapsto a \cdot x$ *is* $([\delta_\mathbb{F}, \delta^=], \delta^=)$*– and* $([\delta_\mathbb{F}, \delta^\geqslant], \delta^\geqslant)$*–computable,*

(5) $0 \in X$ *is a* $\delta^=$*–computable point.*

Proof. The norm is computable w.r.t. $\delta^=$ by definition. By assumption, the limit operation Lim is computable w.r.t. δ. We have to prove that it is computable w.r.t. $\delta^=$ too. In case that $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence with limit x and $||x_i - x_j|| \leq 2^{-j}$ for all $i > j$, we obtain

$$||x|| = \left|\left|\lim_{n\to\infty} x_n\right|\right| = \lim_{n\to\infty} ||x_n||$$

since the norm $|| \ ||$ is continuous and

$$| \ ||x_i|| - ||x_j|| \ | \leq ||x_i - x_j|| \leq 2^{-j}$$

for all $i > j$. Thus, using evaluation and type conversion we can prove that the limit operation Lim is computable w.r.t. $\delta^=$, since it is computable w.r.t. δ and the limit operation on real numbers is computable w.r.t. $\delta_\mathbb{R}$. Analogously, computability of the scalar multiplication w.r.t. $\delta^=$ follows from $||ax|| = |a| \cdot ||x||$ because the scalar multiplication is computable w.r.t. δ and multiplication on the real numbers is computable w.r.t. $\delta_\mathbb{R}$. Analogously, computability of the scalar multiplication w.r.t. $\delta^\geqslant$ follows from $s \geq ||x|| \Longrightarrow |a|s \geq ||ax||$ and computability of addition from

$$s \geq ||x||, t \geq ||y|| \Longrightarrow s + t \geq ||x|| + ||y|| \geq ||x + y||.$$

Finally, the computable points w.r.t. $\delta^=$ are exactly the δ–computable points with computable norm. Especially, $0 \in X$ is δ–computable because of $||0|| = 0$. □

Thus, on the one hand, $\delta^=$ is the optimal representation of X from the topological point of view, since norm and limit operation become computable (and the algebraic operations besides addition are computable too). On the other hand, $\delta^\geqslant$ is the optimal representation of X from the algebraic point of view, since all algebraic operations become computable and additional $\delta^\geqslant$ allows to compute at least upper bounds on the norm of points. Corollary 15.2 yields the following corollary.

Corollary 15.11 *If $(X, || \, ||, \delta)$ is a non-separable general computable normed space, then $\delta^= \not\equiv_t \delta^\geqslant$, the norm $|| \, ||$ is not continuous w.r.t. $\delta^\geqslant$ and vector space addition is not continuous w.r.t. $\delta^=$.*

Thus, in non-separable normed spaces there is a "computational gap" between the algebraic structure and the topological structure of the space. We have already seen in Proposition 15.9 that this gap does not exist in case of ordinary computable normed space. Next we will show that any general computable normed space with $\delta^=$–computable addition which is subject to an additional separability condition gives rise to an ordinary computable normed space.

Definition 15.12 (Effectively separable) A general computable normed space $(X, || \, ||, \delta)$ is called *effectively separable* w.r.t. $e : \mathbb{N} \to X$, if e is a $\delta^=$–computable sequence whose linear span is dense in X.

Since an ordinary computable normed space has as standard representation its Cauchy representation, we can obtain the following stability result.

Theorem 15.13 (Stability Theorem) *If $(X, || \, ||, \delta)$ is a general computable normed space which is effectively separable w.r.t. $e : \mathbb{N} \to X$ and if the vector space addition is computable w.r.t. $\delta^=$, then $(X, || \, ||, e)$ is a computable normed space and $\delta_X \equiv \delta^=$ for the corresponding Cauchy representation δ_X.*

Proof. Let δ_X denote the Cauchy representation of $(X, || \, ||, e)$. We have to prove that the metric space (X, d, α_e) is computable, where d is the metric induced by $|| \, ||$. But computability of $d \circ (\alpha_e \times \alpha_e)$ follows from the fact that e is computable w.r.t. $\delta^=$, from Proposition 15.10, which guarantees that the norm and scalar multiplication is computable w.r.t. $\delta^=$ and from the presumption that the vector space addition is computable w.r.t. $\delta^=$. Especially, this implies that d is computable w.r.t. δ_X; it is easy to see that with the help of e one can construct a computable δ_X–name of 0 and thus the norm is computable w.r.t. δ_X too. Moreover, the limit operation is computable w.r.t. $\delta^=$ by Proposition 15.10 and thus one can directly prove $\delta_X \leq \delta^=$. Since the metric d is $([\delta^=, \delta^=], \delta_\mathbb{R})$–computable, it follows that d is also $([\delta^=, \delta_X], \delta_\mathbb{R})$–computable and thus $\delta^= \leq \delta_X$ by Proposition 3.2(1). Altogether, this implies $\delta_X \equiv \delta^=$ and this implies again by Proposition 15.10 and by presumption that (X, δ_X) is a computable vector space. □

This theorem is closely related to Pour-El and Richards Stability Lemma [PER89] which is a special case of more general stability results for topological structures [Bra99c, Bra99b]. In the following we will show that general computable normed spaces fulfill some closure properties of normed spaces. Besides subspace and product space construction we will also prove that the dual space of an ordinary computable normed space is a general computable normed space. We start with the subspace construction.

Proposition 15.14 (Subspace) *If $(X, ||\ ||, \delta)$ is a general computable normed space with a linear subspace $Y \subseteq X$, then the subspace $(Y, ||\ ||_Y, \delta|^Y)$ is a general computable normed space too and the canonical injection $Y \hookrightarrow X$ is computable. Moreover, $\delta^=|^Y = \delta|^{Y=}$ and $\delta^{\geqslant}|^Y = \delta|^{Y\geqslant}$.*

Here $||\ ||_Y$ denotes the restriction of $||\ ||$ to Y in the source and $\delta|^Y$ denotes the restriction of δ to Y in the target. The proof follows from the fact that all subspace operations, including the limit, are defined as restriction. We proceed with the product space construction.

Proposition 15.15 (Product space) *If $(X, ||\ ||_X, \delta_X)$ and $(Y, ||\ ||_Y, \delta_Y)$ are general computable normed spaces, then the product space $(X \times Y, ||\ ||, [\delta_X, \delta_Y])$ with $||(x,y)|| := ||x||_X + ||y||_Y$ is a general computable normed space too. The canonical projections $\mathrm{pr}_1 : X \times Y \to X$ and $\mathrm{pr}_2 : X \times Y \to Y$ are computable. Moreover, $[\delta_X^=, \delta_Y^=] \leq [\delta_X, \delta_Y]^=$ and $[\delta_X^{\geqslant}, \delta_Y^{\geqslant}] \equiv [\delta_X, \delta_Y]^{\geqslant}$.*

Proof. By assumption, (X, δ_X) and (Y, δ_Y) are computable vector spaces with computable limit operations. Since the vector space operations are defined componentwise, it follows that $(X \times Y, [\delta_X, \delta_Y])$ is a computable vector space too. Now let $(x_n, y_n)_{n \in \mathbb{N}}$ be a rapidly converging sequence in $X \times Y$, i.e. $||(x_i, y_i) - (x_j, y_j)|| = \max\{||x_i - x_j||_X, ||y_i - y_j||_Y\} \leq 2^{-j}$ for all $i > j$. It follows $||x_i - x_j||_X \leq 2^{-j}$ and $||y_i - y_j||_X \leq 2^{-j}$ for all $i > j$. Thus, the limit operation of $(X \times Y, ||\ ||)$ is also computable w.r.t. $[\delta_X, \delta_Y]$ since it suffices to compute the limit componentwise. The projections are obviously computable. We still have to prove the statements on reducibility. Since $\max : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is computable, there exists a computable realization $G :\subseteq \Sigma^\omega \to \Sigma^\omega$. We define $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ by $F\langle\langle p, q\rangle, \langle r, s\rangle\rangle := \langle\langle p, r\rangle, G\langle q, s\rangle\rangle$. Then F is computable and we obtain

$$
\begin{aligned}
& [\delta_X^=, \delta_Y^=]\langle\langle p, q\rangle, \langle r, s\rangle\rangle = (\delta_X^=\langle p, q\rangle, \delta_Y^=\langle r, s\rangle) = (x, y) \\
\Longrightarrow\ & \delta_X(p) = x, \delta_{\mathbb{R}}(q) = ||x||_X \text{ and } \delta_Y(r) = y, \delta_{\mathbb{R}}(s) = ||y||_Y \\
\Longrightarrow\ & [\delta_X, \delta_Y]\langle p, r\rangle = (x, y) \text{ and } \delta_{\mathbb{R}} G\langle q, s\rangle = \max\{||x||_X, ||y||_Y\} = ||(x, y)|| \\
\Longrightarrow\ & [\delta_X, \delta_Y]^= F\langle\langle p, q\rangle, \langle r, s\rangle\rangle = [\delta_X, \delta_Y]^=\langle\langle p, r\rangle, G\langle q, s\rangle\rangle = (x, y).
\end{aligned}
$$

Hence, $[\delta_X^=, \delta_Y^=] \leq [\delta_X, \delta_Y]^=$. Analogously, one can prove $[\delta_X^{\geqslant}, \delta_Y^{\geqslant}] \leq [\delta_X, \delta_Y]^{\geqslant}$. On the other hand, define $H :\subseteq \Sigma^\omega \to \Sigma^\omega$ by $H\langle\langle p, r\rangle, t\rangle := \langle\langle p, t\rangle, \langle r, t\rangle\rangle$.

Then H is computable and we obtain

$$\begin{aligned}
& [\delta_X,\delta_Y]^{\geqslant}\langle\langle p,r\rangle,t\rangle = (x,y)\\
\Longrightarrow\ & [\delta_X,\delta_Y]\langle p,r\rangle = (x,y) \text{ and } \delta_{\mathbb{R}}(t) \geq ||(x,y)|| = \max\{||x||_X, ||y||_Y\}\\
\Longrightarrow\ & \delta_X(p) = x, \delta_Y(r) = y \text{ and } \delta_{\mathbb{R}}(t) \geq ||x||_X, \delta_{\mathbb{R}}(t) \geq ||y||_Y\\
\Longrightarrow\ & [\delta_X^{\geqslant},\delta_Y^{\geqslant}]H\langle\langle p,r\rangle,t\rangle = [\delta_X^{\geqslant},\delta_Y^{\geqslant}]\langle\langle p,t\rangle,\langle r,t\rangle\rangle = (x,y).
\end{aligned}$$

Hence, $[\delta_X,\delta_Y]^{\geqslant} \leq [\delta_X^{=},\delta_Y^{=}]$. □

Now we will discuss operator spaces. Even the dual space $X' = \mathcal{B}(X,\mathbb{F})$ of a separable normed space X is not necessarily separable. Thus it follows that computable normed spaces are not closed under dual space construction. However, the operator space $\mathcal{B}(X,Y)$ of computable normed spaces X,Y is at least a general computable normed space as the following result shows.

Proposition 15.16 (Operator space) *If X,Y are computable normed spaces with Cauchy representations δ_X,δ_Y, then the space of linear bounded operators $(\mathcal{B}(X,Y),||\ ||,\delta)$ with the operator norm $||T|| := \sup_{||x||=1}||Tx||$ and representation $\delta := [\delta_X \to \delta_Y]|^{\mathcal{B}(X,Y)}$ is a general computable Banach space. Moreover, $\delta_{\mathcal{B}(X,Y)} = \delta^{=}$.*

Proof. We have to prove that the space $(\mathcal{B}(X,Y),\delta)$ is a computable vector space with a computable limit operation $\mathrm{Lim} :\subseteq \mathcal{B}(X,Y)^{\mathbb{N}} \to \mathcal{B}(X,Y)$. Since vector space addition $+ : Y \times Y \to Y, (x,y) \mapsto x+y$ and scalar multiplication $\cdot : \mathbb{F} \times Y \to Y, (a,y) \mapsto a \cdot y$ are computable w.r.t. δ_Y, it follows by evaluation and type conversion that the operator vector space addition $+ : \mathcal{B}(X,Y) \times \mathcal{B}(X,Y) \to \mathcal{B}(X,Y), (f,g) \mapsto f+g$ and scalar multiplication $\cdot : \mathbb{F} \times \mathcal{B}(X,Y) \to \mathcal{B}(X,Y), (a,f) \mapsto a \cdot f$ are computable w.r.t. $[\delta_X \to \delta_Y]$ too. Since $0 \in Y$ is a δ_Y–computable point, the zero function $z \in \mathcal{B}(X,Y)$ is a computable point too. Now let us assume that $(T_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{B}(X,Y)$ which converges to some $T \in \mathcal{B}(X,Y)$ such that $||T_i - T_j|| \leq 2^{-j}$ for all $i > j$. Then $||T_i x - T_j x|| \leq ||T_i - T_j|| \cdot ||x|| \leq 2^{-j}||x||$. Given $x \in X$ w.r.t. δ_X, we can effectively find some $k \in \mathbb{N}$ such that $||x|| \leq 2^k$ and thus $||T_{i+k}x - T_{j+k}x|| \leq 2^{-j-k}||x|| \leq 2^{-j}$ and $\lim_{i\to\infty} T_{i+k}x = Tx$. Since $\mathrm{Lim}_Y :\subseteq Y^{\mathbb{N}} \to Y$ is computable w.r.t. δ_Y, we can effectively evaluate the limit operation $\mathrm{Lim} :\subseteq \mathcal{B}(X,Y)^{\mathbb{N}} \to \mathcal{B}(X,Y)$. By type conversion it follows that Lim is computable w.r.t. $[\delta_X \to \delta_Y]$ too. □

The same proof would work in case of general computable normed spaces (X,δ_X) and (Y,δ_Y) with $\delta = [\delta_X^{\geqslant} \to \delta_Y^{\geqslant}]$. However, in this case it is not clear which space is represented by δ. If $\delta^{\geqslant}$ is an admissible representation of an admissible topology, i.e. a topology which makes all linear functionals $f : X \to \mathbb{F}$ continuous, then $\delta = [\delta_X^{\geqslant} \to \delta_{\mathbb{F}}]$ could at least be restricted to a dual space representation. But it deserves further investigations to study this question. Here, we only formulate a simple corollary of the previous result.

Corollary 15.17 (Dual space) *If X is a computable normed space with Cauchy representation δ, then the dual space $(X', ||\ ||', \delta')$ with $X' = \mathcal{B}(X, \mathbb{F})$, the operator norm $||f||' := \sup_{||x||=1} |f(x)|$ and $\delta' := [\delta \to \delta_{\mathbb{F}}]|^{X'}$ is a general computable Banach space. Moreover, $\delta_{X'} = \delta'^{=}$.*

As a last example of a closure property we discuss the space of bounded sequences. The proof is a simplified version of the previous proof.

Proposition 15.18 (Bounded sequences) *If (Y, δ_Y) is a general computable normed space, then the space of bounded sequences $(\mathcal{B}(\mathbb{N}, Y), ||\ ||, \delta)$ with the supremum norm $||(y_n)_{n\in\mathbb{N}}|| := \sup_{n\in\mathbb{N}} ||y_n||$ and $\delta := \delta_Y^{\mathbb{N}}|^{\mathcal{B}(\mathbb{N},Y)}$ is a general computable normed space. Moreover, $\delta_{\mathcal{B}(\mathbb{N},Y)} = \delta^{=}$.*

Proof. We have to prove that the space $(\mathcal{B}(\mathbb{N}, Y), ||\ ||, \delta)$ is a computable vector space with a computable limit operation $\mathrm{Lim} :\subseteq \mathcal{B}(\mathbb{N}, Y)^{\mathbb{N}} \to \mathcal{B}(\mathbb{N}, Y)$. Since vector space addition $+ : Y \times Y \to Y, (x, y) \mapsto x + y$ and scalar multiplication $\cdot : \mathbb{F} \times Y \to Y, (a, y) \mapsto a \cdot y$ are computable w.r.t. δ_Y, it follows by evaluation and type conversion that the sequence vector space addition $+ : \mathcal{B}(\mathbb{N}, Y) \times \mathcal{B}(\mathbb{N}, Y) \to \mathcal{B}(\mathbb{N}, Y), (f, g) \mapsto f + g$ and scalar multiplication $\cdot : \mathbb{F} \times \mathcal{B}(\mathbb{N}, Y) \to \mathcal{B}(\mathbb{N}, Y), (a, f) \mapsto a \cdot f$ are computable too w.r.t. $\delta_Y^{\mathbb{N}} = [\delta_{\mathbb{N}} \to \delta_Y]$. Since $0 \in Y$ is a δ_Y–computable point, the zero sequence $z \in \mathcal{B}(\mathbb{N}, Y)$ is a computable point too. Now let us assume that $(y_n)_{n\in\mathbb{N}} = ((y_{kn})_{k\in\mathbb{N}})_{n\in\mathbb{N}}$ is a sequence in $\mathcal{B}(\mathbb{N}, Y)$ which converges to some $z = (z_k)_{k\in\mathbb{N}} \in \mathcal{B}(\mathbb{N}, Y)$ such that $||y_i - y_j|| \le 2^{-j}$ for all $i > j$. Then $||y_{ki} - y_{kj}|| \le \sup_{k\in\mathbb{N}} ||y_{ki} - y_{kj}|| = ||y_i - y_j|| \le 2^{-j}$ for all $i > j$ and thus $\lim_{i\to\infty} y_{ki} = z_k$ for all $k \in \mathbb{N}$. Since $\mathrm{Lim}_Y :\subseteq Y^{\mathbb{N}} \to Y$ is computable w.r.t. δ_Y, we can effectively evaluate the limit operation $\mathrm{Lim} :\subseteq \mathcal{B}(\mathbb{N}, Y)^{\mathbb{N}} \to \mathcal{B}(\mathbb{N}, Y)$. By type conversion it follows that Lim is computable w.r.t. $\delta_Y^{\mathbb{N}} = [\delta_{\mathbb{N}} \to \delta_Y]$ too. □

Especially, we obtain the following corollary on the space ℓ_∞.

Corollary 15.19 *(ℓ_∞, δ) with $\delta = \delta_{\mathbb{F}}^{\mathbb{N}}|^{\ell_\infty}$ is a general computable normed space and $\delta_{\ell_\infty} = \delta^{=}$.*

We proceed with a brief discussion of admissibility properties of representations of general computable normed spaces. We directly can conclude the following result on admissibility with the help of closure properties proved by Schröder (in Section 4.2 and 4.5 of [Sch00]).

Theorem 15.20 (Admissibility) *Let $(X, ||\ ||, \delta)$ be a general computable normed space and let δ be an admissible representation of X with respect to some T_1–topology τ. Then*

(1) $\delta^{=}$ is admissible with respect to the weakest topology $\tau^{=}$ such that the identity $\mathrm{id} : (X, \tau^{=}) \hookrightarrow (X, \tau)$ and the norm $||\ || : (X, \tau^{=}) \to \mathbb{R}$ become continuous,

(2) $\delta^{\geqslant}$ is admissible with respect to the inductive limit topology $\tau^{\geqslant} = \underrightarrow{\lim}\, \tau_k$ of the subtopologies τ_k of τ on $X_k := \{x \in X : ||x|| \le k\}$ for all $k \in \mathbb{N}$.

Here $X = \bigcup_{k=0}^{\infty} X_k$ and $\underrightarrow{\lim}\, \tau_k := \{U \subseteq X : (\forall k)\ U \cap X_k \in \tau_k\}$. If, in addition to the assumptions of the theorem, $\tau_{||\ ||} \supseteq \tau$ holds for the norm topology $\tau_{||\ ||}$ of X, then we obtain

$$\tau_{||\ ||} \supseteq \tau^{=} \supseteq \tau^{\geqslant} \supseteq \tau.$$

In view of Proposition 15.1 neither $\delta^{=}$ nor $\delta^{\geqslant}$ can be admissible w.r.t. the norm topology $\tau_{||\ ||}$ in case of a non-separable general computable normed space $(X, ||\ ||, \delta)$. Thus, the previous statement expresses in a certain sense the best what we can expect. Especially, our representations $\delta_{\ell_\infty}, \delta_{\mathcal{B}(X,Y)}$ and $\delta_{\mathcal{B}(\mathbb{N},X)}$, defined for computable normed spaces X, Y, are all admissible w.r.t. the corresponding topologies $\tau^{=}$.

However, we still have to justify our ad hoc choice of representations. $\delta_{\ell_\infty}, \delta_{\mathcal{B}(X,Y)}$ and $\delta_{\mathcal{B}(\mathbb{N},X)}$. All these representations are of type $\delta^{=}$ and not of type $\delta^{\geqslant}$. One such justification can be seen in the computable version of Landau's Theorem (i.e. Theorem 13.3 and Corollary 13.4). We reformulate these results in terms of $\delta^{=}$ and $\delta^{\geqslant}$.

Corollary 15.21 (Computable Theorem of Landau) *Let $p, q > 1$ be computable real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ or $p = 1$ and $q = \infty$. And let $\delta_p := \delta_{\mathbb{F}}^{\mathbb{N}}|^{\ell_p}$ and $\delta'_p := [\delta_p^{=} \to \delta_{\mathbb{F}}]|^{\ell'_p}$.*

(1) $(\ell_q, \delta_q^{=})$ is computably isometric isomorphic to $(\ell'_p, \delta_p'^{=})$.

(2) $(\ell_q, \delta_q^{\geqslant})$ is computably isometric isomorphic to $(\ell'_p, \delta_p'^{\geqslant})$.

If we accept our choice of $\delta_{\ell_q} \equiv \delta_q^{=}$ as standard representation of ℓ_q for real numbers $q \ge 1$ (and actually there is not much doubt that something could be wrong with this), then (1) leads naturally to $\delta_{\ell'_p} \equiv \delta_p'^{=}$ as standard representation of ℓ'_p for real numbers $p > 1$. Then it is natural to take the corresponding representation also in case $p = 1$ and in this case (1) leads to $\delta_{\ell_\infty} \equiv \delta_\infty$ as natural standard representation for ℓ_∞. In other words: if we want to keep the natural computable isometric isomorphism results of Corollary 13.4, then there is no alternative to our choice of the representation of ℓ_∞ and of the dual space representations of ℓ'_p (up to computable equivalence).

On the other hand, as we will see next, our choice of a standard representation of ℓ_∞ makes it impossible to extend the computable versions of Banach's Inverse Mapping Theorem and the Closed Graph Theorem to the non-separable case. Again we will use diagonal operators to construct counterexamples. Therefore we transfer Proposition 5.7 to the non-separable case. By $\ell_\infty^=, \ell_\infty^\geqslant$ we denote the represented spaces $(\ell_\infty, \delta^=)$, $(\ell_\infty, \delta^\geqslant)$ with $\delta := \delta_\mathbb{F}^\mathbb{N}|^{\ell_\infty}$ and endowed with the corresponding final topologies $\tau^=, \tau^\geqslant$, respectively (see above).

Proposition 15.22 (Diagonal operator) *There exists a computable multi-valued operation $\tau :\subseteq \mathbb{R}_> \rightrightarrows \mathcal{C}(\ell_\infty^=, \ell_\infty^=)$ such that for any $a \in \mathbb{R}_>$ with $a \in (0,1]$ there exists some $T_a \in \tau(a)$ and all such $T_a : \ell_\infty \to \ell_\infty$ are diagonal operators of a. The same holds true with $\ell_\infty^\geqslant$ instead of $\ell_\infty^=$.*

Proof. Given a real number $a \in \mathbb{R}_>$ with $a \in (0,1]$, we can effectively find a decreasing sequence $(a_n)_{n\in\mathbb{N}}$ of rational numbers $a_n \in \mathbb{Q}$ such that $a_0 = 1$ and $a = \inf_{n\in\mathbb{N}} a_n$. We define a diagonal operator $T_a : \ell_\infty \to \ell_\infty$ of a by $T_a(x_k)_{k\in\mathbb{N}} := (a_k x_k)_{k\in\mathbb{N}}$ for all $(x_k)_{k\in\mathbb{N}} \in \ell_\infty$. Let $\delta := \delta_\mathbb{F}^\mathbb{N}|^{\ell_\infty}$. Using evaluation and type conversion it follows that T_a can be effectively evaluated w.r.t. δ. Thus, there exists some $(\delta_\mathbb{R}^>, [\delta \to \delta])$–computable τ with the desired property. Since

$$||T_a x|| \leq ||T_a|| \cdot ||x|| = ||x||$$

we directly obtain that there also exists a $(\delta_\mathbb{R}^>, [\delta^\geqslant \to \delta^\geqslant])$–computable τ. We still have to prove that there also exists some $(\delta_\mathbb{R}^>, [\delta^= \to \delta^=])$–computable τ. Given some $x = (x_k)_{k\in\mathbb{N}} \in \ell_\infty$ w.r.t. $\delta^=$ and a precision $m \in \mathbb{N}$, we can effectively find some $n \in \mathbb{N}$ such that $|\,|x_n| - ||x||\,| < 2^{-m}$. Hence, we can compute $s := \max_{k=0,\dots,n} |a_k x_k|$ and it follows

$$|s - ||T_a x||\,| = \left| \max_{k=0,\dots,n} |a_k x_k| - \sup_{k\in\mathbb{N}} |a_k x_k| \right| < 2^{-m}$$

since $(a_k)_{k\in\mathbb{N}}$ is decreasing with $a_k \leq 1$ for all $k \in \mathbb{N}$. Thus, we can also evaluate T_a w.r.t. $\delta^=$ up to any given precision m. Using type conversion we can actually prove that there exists an operation τ with the desired properties. □

As a consequence we obtain the following corollary which yields some limitations to possible computable versions of Banach's Inverse Mapping Theorem and the Closed Graph Theorem for non-separable spaces.

Corollary 15.23 *Let $a \in (0,1]$ be some left-computable real number which is not right-computable. Then there exists a diagonal operator $T_a : \ell_\infty \to \ell_\infty$ of a which is $(\delta_{\ell_\infty}, \delta_{\ell_\infty})$–computable but T_a^{-1} cannot be $(\delta_{\ell_\infty}, \delta_{\ell_\infty})$–computable since $||T_a^{-1}e|| = \frac{1}{a}$ for the δ_{ℓ_∞}–computable point $e = (1,1,1,...)$.*

However, it is easy to see that this counterexample cannot be transfered to $\delta^{\geqslant}$ instead of $\delta^{=} \equiv \delta_{\ell_\infty}$ (with δ as above) since the inverse T_a^{-1} of a $(\delta^{\geqslant}, \delta^{\geqslant})$–computable diagonal operator T_a is always $(\delta^{\geqslant}, \delta^{\geqslant})$–computable too. We leave it to further investigations to continue the study of non-separable spaces.

16 Conclusion

We will close this paper with a short survey on some of our results and a brief discussion of related results in constructive analysis and reverse mathematics. We briefly resume our results on the Open Mapping Theorem, Banach's Inverse Mapping Theorem and the Closed Graph Theorem, including the finite-dimensional and the non-separable case. Let us assume for the following that X, Y are computable Banach spaces. Then our results on the Open Mapping Theorem can be summarized as shown in Table 1.

$T : X \to Y$ linear bounded and surjective	finite-dimensional Y	general case
T computable, U r.e. open $\Longrightarrow T(U)$ r.e. open	$+$	$+$
T computable $\Longrightarrow U \mapsto T(U)$ computable	$+$	$+$
$T \mapsto (U \mapsto T(U))$ computable	$+$	$-$

Table 1: Computable versions of the Open Mapping Theorem

Each "+" indicates a positive result and each "−" indicates a negative result or more precisely, it indicates that the corresponding property is not fulfilled in general in the given case. The results in the first and second row are based on Corollary 5.4. The positive result for the finite-dimensional case in the third row is based on Theorem 14.3 and the negative result for the general case is based on Theorem 5.8 (or on Corollary 5.10, alternatively).

Analogously, we summarize our results on the Inverse Mapping Theorem in Table 2. Here we can also include the non-separable case, provided that we consider X, Y as general computable Banach spaces represented by their corresponding $\delta^{=}$ representations, as defined in Section 15. The negative results for the non-separable case in the third column are based on Corollary 15.23. The positive results in the first row are based on Corollary 6.3. The positive result for the finite-dimensional case in the second row is based on Corollary 14.4

$T : X \to Y$ lin. bounded and bijective	finite-dimensional	separable	non-separable
T computable $\Longrightarrow T^{-1}$ computable	+	+	−
$T \mapsto T^{-1}$ computable	+	−	−

Table 2: Computable versions of Banach's Inverse Mapping Theorem

and the negative result for the separable case is based on Corollary 6.5 (or on Corollary 6.7, alternatively). Finally, we summarize the results on computable versions of the Closed Graph Theorem in Table 3.

$T : X \to Y$ linear bounded	finite-dimensional Y	general case
graph(T) recursive closed $\Longrightarrow T$ computable	+	+
graph(T) $\mapsto T$ computable	+	−

Table 3: Computable versions of the Closed Graph Theorem

The results in the first row are based on Theorem 8.7. The positive result for the finite-dimensional case in the second row is based on Theorem 14.6 and the negative result for the general case is based on Theorem 8.5.

We emphasize again that all our results are based on the following point of view of computable analysis: computability, described by Turing machines, is considered as a further classical property of ordinary objects (points, functions, spaces and so on). Especially, we can use the law of excluded middle and any other classical laws of reasoning without any restrictions.

Related Results in Constructive Analysis

In Bishop's school of constructive analysis [BB85, Bri79][4] the principles of Banach spaces have already been studied intensively [BR87, BJM89, Ish94, Ish97, BI98, BI01]. The underlying philosophy is somehow to develop "ordinary analysis" but with intuitionistic logic (that is essentially without the law of excluded middle). Roughly speaking, an implication arrow "$\Longrightarrow$" in the intuitionistic setting of constructive analysis can be replaced by a "constructive mapping" arrow "$\mapsto$". Thus, the distinctions which we have made in Tables 1, 2, 3, cannot be expressed in constructive analysis in the same way. However, there are close relations between our results and results which have been obtained in constructive analysis. On the one hand, it is likely that our negative results (like the statement that $T \mapsto T^{-1}$ is not computable under certain

[4]For a different approach to constructive functional analysis, based on the ideas of Lorenzen and Weyl, see [Zah78].

assumptions) imply that the corresponding theorems cannot be proved in the constructive setting (cf. [Tro92] for a partial transfer result). On the other hand, theorems which can be proved in the constructive setting plus BD-$\mathbb{N}$ (a principle which states that each "pseudobounded" subset of $\mathbb{N}$ is bounded) are valid in Markov's school of constructive analysis [Kuš84] where only computable objects are considered. Thus, it seems that theorems which have been established in Bishop's style plus BD-$\mathbb{N}$ imply weak computable theorems in our sense (like the statement that T computable $\Longrightarrow T^{-1}$ computable). In this respect, it is interesting to note that Ishihara proved versions of Banach's Inverse Mapping Theorem, the Open Mapping Theorem and the Closed Graph Theorem based on the notion of sequential continuity (which is constructively weaker than continuity) [Ish94, Ish97]. Since the statement that any sequential continuous mapping (between separable metric spaces) is continuous, is equivalent to the principle BD-$\mathbb{N}$ (cf. Theorem 4 in [Ish92]), it follows that Ishihara's results are closely related to our weak computable results. In turn, our negative weak results (like the statement that there exists a computable sequence (T_i) of operators such that (T_i^{-1}) is not a computable sequence) should imply that the corresponding statements cannot even be derived in the constructive setting plus BD-$\mathbb{N}$. To establish the precise relation between computable and constructive analysis remains an interesting open problem for future investigations.

Related Results in Reverse Analysis

In reverse mathematics, as proposed by Friedman and Simpson [Sim99], several principles of Banach spaces have been analysed according to which axioms are needed to prove the corresponding theorems in the language of second order arithmetic. Especially, it has been shown that the subsystem RCA_0 of second order arithmetic (i.e. second order arithmetic with a restricted recursive comprehension axiom) suffices to prove the Baire Category Theorem [BS93, Sim99, Bro87]. This setting rather corresponds to Markov's constructive analysis [Kuš84] where only computable objects are considered. Maybe a kind of higher order reverse mathematics might come closer to the uniform point of view of computable analysis [Koh]. Moreover, the proofs in reverse mathematics are not necessarily effective such that they cannot be directly transfered to computable analysis (an example is the proof of the Banach-Steinhaus Theorem II.10.8 in [Sim99], where the Baire Category Theorem is applied in the non-effective contrapositive version). However, computable counterexamples, provided by computable analysis, could be a relevant source for reverse mathematics (cf. the discussion in Remark I.8.5 of [Sim99]).

A Representations of the Operator Space

In this appendix we will discuss several representations of the space $\mathcal{B}(X,Y)$ of linear bounded operators $T : X \to Y$, which have implicitly been considered in the previous sections. We will only consider ordinary computable normed spaces X, Y.

Definition A.1 (Representations of the operator space) Let X and $(Y, || \, ||, e)$ be computable normed spaces. We define representations of $\mathcal{B}(X,Y)$ as follows:

(1) $\delta_{\text{ev}}(p) = T :\iff [\delta_X \to \delta_Y](p) = T$,

(2) $\delta_{\text{graph}}(p) = T :\iff \delta_{\mathcal{A}(X \times Y)}(p) = \text{graph}(T)$,

(3) $\delta^{<}_{\text{graph}}(p) = T :\iff \delta^{<}_{\mathcal{A}(X \times Y)}(p) = \text{graph}(T)$,

(4) $\delta_{\text{seq}}(p) = T :\iff \delta^{\mathbb{N}}_{Y}(p) = (Te_i)_{i \in \mathbb{N}}$,

for all $p \in \Sigma^\omega$ and linear bounded operators $T : X \to Y$.

Besides the mentioned representations we will also consider those variants that are obtained by adding information on the operator bound $||T|| = \sup_{||x||=1} ||Tx||$ of the represented operator $T : X \to Y$, as defined in the Section 15. For completeness we repeat the definition: if δ is a representation of $\mathcal{B}(X,Y)$, then we define representations $\delta^{=}, \delta^{\geqslant}$ of $\mathcal{B}(X,Y)$ by

(1) $\delta^{=}\langle p, q\rangle = T :\iff \delta(p) = T$ and $\delta_{\mathbb{R}}(q) = ||T||$,

(2) $\delta^{\geqslant}\langle p, q\rangle = T :\iff \delta(p) = T$ and $\delta_{\mathbb{R}}(q) \geq ||T||$.

Now we can reformulate several of our theorems as reducibility results on the representations given above. The next theorems especially includes a uniform version of Theorem 8.7.

Theorem A.2 (Representations of the operator space) *Let X and Y be computable Banach spaces. Then the following reducibilities for representations of $\mathcal{B}(X,Y)$ hold:*

(1) $\delta^{=}_{\text{ev}} \leq \delta_{\text{ev}} \leq \delta_{\text{seq}} \leq \delta^{<}_{\text{graph}}$ *and* $\delta_{\text{ev}} \leq \delta_{\text{graph}} \leq \delta^{<}_{\text{graph}}$,

(2) $\delta_{\text{ev}} \equiv \delta^{\geqslant}_{\text{ev}} \equiv \delta^{\geqslant}_{\text{seq}} \equiv \delta^{<\geqslant}_{\text{graph}} \equiv \delta^{\geqslant}_{\text{graph}}$,

(3) $\delta^{=}_{\text{ev}} \equiv \delta^{=}_{\text{seq}} \equiv \delta^{<=}_{\text{graph}} \equiv \delta^{=}_{\text{graph}}$.

Proof. (1) "$\delta_{\mathrm{ev}}^{=} \leq \delta_{\mathrm{ev}}$" holds obviously and "$\delta_{\mathrm{ev}} \leq \delta_{\mathrm{seq}}$" follows by the evaluation property.
"$\delta_{\mathrm{seq}} \leq \delta_{\mathrm{graph}}^{<}$" can be proved similar as "(3)$\Longrightarrow$(4)" of Theorem 8.7. Therefore let $T : X \to Y$ be a linear bounded operator and let $(y_n)_{n\in\mathbb{N}}$ be the sequence in Y with $y_i := Te_i$ for all $i \in \mathbb{N}$. Let $\alpha_e\langle k, \langle n_0, ..., n_k\rangle\rangle := \sum_{i=0}^{k} \alpha_{\mathbb{F}}(n_i)e_i$. By linearity of T it follows

$$T\alpha_e\langle k, \langle n_0, ..., n_k\rangle\rangle = T\left(\sum_{i=0}^{k} \alpha_{\mathbb{F}}(n_i)e_i\right) = \sum_{i=0}^{k} \alpha_{\mathbb{F}}(n_i)y_i.$$

Thus, given $(y_n)_{n\in\mathbb{N}}$ we can compute $T\alpha_e$ since the algebraic operations in Y are computable. Using type conversion we can effectively find the sequence $f : \mathbb{N} \to X \times Y, i \mapsto (\alpha_e(i), T\alpha_e(i))$ which is dense in graph(T) by continuity of T, since α_e is dense in X. By Proposition 4.5 the desired result follows.
"$\delta_{\mathrm{ev}} \leq \delta_{\mathrm{graph}}$" follows from the Graph Theorem 8.3. "$\delta_{\mathrm{graph}} \leq \delta_{\mathrm{graph}}^{<}$" follows from $\delta_{\mathcal{A}(X\times Y)} \leq \delta_{\mathcal{A}(X\times Y)}^{<}$.

(2) "$\delta_{\mathrm{ev}} \leq \delta_{\mathrm{ev}}^{\gtrless}$" follows from Theorem 9.10 and "$\delta_{\mathrm{ev}}^{\gtrless} \leq \delta_{\mathrm{ev}}$" holds obviously.
"$\delta_{\mathrm{ev}}^{\gtrless} \leq \delta_{\mathrm{seq}}^{\gtrless} \leq \delta_{\mathrm{graph}}^{<\gtrless}$" follows directly from (1).
"$\delta_{\mathrm{graph}}^{<\gtrless} \leq \delta_{\mathrm{ev}}^{\gtrless}$" Given graph$(T) \in \mathcal{A}_{<}(X \times Y)$ of some linear bounded operator $T : X \to Y$, we can effectively find a sequence $f : \mathbb{N} \to X \times Y$ such that range(f) is dense in graph(T) by Proposition 4.5. By Proposition 3.11 we can effectively find the projections $f_1 : \mathbb{N} \to X$ and $f_2 : \mathbb{N} \to Y$ of f too. Given $x \in X$, a real number $s \geq ||T||$ and a precision $m \in \mathbb{N}$ we can effectively find find some $n \in \mathbb{N}$ and numbers $q_0, ..., q_n \in Q_{\mathbb{F}}$ such that $||\sum_{i=0}^{n} q_i f_1(i) - x|| < \frac{1}{s}2^{-m}$ since range(f_1) is dense in X. It follows

$$\left|\left|T\left(\sum_{i=0}^{n} q_i f_1(i)\right) - T(x)\right|\right| \leq ||T|| \cdot \left|\left|\sum_{i=0}^{n} q_i f_1(i) - x\right|\right| < 2^{-m}.$$

By linearity of T we obtain $T(\sum_{i=0}^{n} q_i f_1(i)) = \sum_{i=0}^{n} q_i T f_1(i) = \sum_{i=0}^{n} q_i f_2(i)$ and thus we can evaluate T effectively up to any given precision m. Using type conversion we obtain the desired reducibility.
"$\delta_{\mathrm{ev}}^{\gtrless} \leq \delta_{\mathrm{graph}}^{\gtrless}$" and "$\delta_{\mathrm{graph}}^{\gtrless} \leq \delta_{\mathrm{graph}}^{<\gtrless}$" follow from (1).

(3) This directly follows from (2). □

It should be mentioned that we have used completeness only for the reduction "$\delta_{\mathrm{graph}}^{<\gtrless} \leq \delta_{\mathrm{ev}}^{\gtrless}$" (for the application of Proposition 4.5) and thus for the results in (2) and (3), while the statement of (1) remains true for computable normed spaces X, Y. In case of a finite-dimensional $Y = \mathbb{F}^m$ we obtain one further equivalence as a direct consequence of Theorem 14.6 since all normed spaces X are everywhere connected.

Corollary A.3 *Let X be a computable normed space and $m \geq 1$ a natural number. Then $\delta_{\text{ev}} \equiv \delta_{\text{graph}}$ for the corresponding representations of $\mathcal{B}(X, \mathbb{F}^m)$.*

Now the question appears which of the reducibilities given in Theorem A.2(1) are strict reducibilities. At least for certain spaces all of these reducibilities are strict as the following theorem shows. Figure 2 summarizes the results.

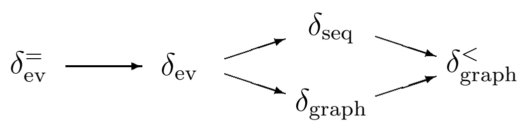

Figure 2: Representations of the operator space $\mathcal{B}(\ell_p, \ell_p)$.

Here an arrow means $\leq$ and $\not\geq_t$. The transitive closure of the diagram is complete, i.e. all missing arrows in the closure indicate $\not\leq_t$.

Theorem A.4 *Let $p \geq 1$ be some computable real number. Then the following holds for representations of the space $\mathcal{B}(\ell_p, \ell_p)$ of linear bounded operators:*

(1) $\delta_{\text{graph}}^{<} \not\leq_t \delta_{\text{graph}} \not\leq_t \delta_{\text{seq}} \not\leq_t \delta_{\text{ev}} \not\leq_t \delta_{\text{ev}}^{=}$,

(2) $\delta_{\text{graph}}^{<} \not\leq_t \delta_{\text{seq}} \not\leq_t \delta_{\text{graph}} \not\leq_t \delta_{\text{ev}}$.

In case of the operator space $\mathcal{B}(\ell_p, \mathbb{F}^m)$ with $m \geq 1$ the same statements hold with the exception of $\delta_{\text{graph}} \not\leq_t \delta_{\text{seq}}$ and $\delta_{\text{graph}} \not\leq_t \delta_{\text{ev}}$.

Proof. For the first two results we only have to consider the case $\mathcal{B}(\ell_p, \ell_p)$.

"$\delta_{\text{graph}} \not\leq_t \delta_{\text{ev}}$" follows from Theorem 8.5.

"$\delta_{\text{graph}} \not\leq_t \delta_{\text{seq}}$" Let us assume that $\delta_{\text{graph}} \leq_t \delta_{\text{seq}}$ is realized by a continuous function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$. Let us consider the zero operator $T : \ell_p \to \ell_p$, i.e. $Tx = 0$ for all $x \in \ell_p$. Given a sequence $q \in \text{dom}(\delta_{\text{graph}})$ with $\delta_{\text{graph}}(q) = T$ the function F produces a sequence $t = F(q)$ such that $\delta_{\text{seq}}(t) = T$. Moreover, the mapping $T \mapsto Te_0$ is obviously $(\delta_{\text{seq}}, \ell_p)$–continuous and realized by some continuous function $G :\subseteq \Sigma^\omega \to \Sigma^\omega$. We consider $H := G \circ F$. Since H is continuous too and $\delta_{\ell_p} H(q) = Te_0 = 0$, there is some finite prefix w of q such that $\delta_{\ell_p} H(w\Sigma^\omega) \subseteq B(0,1)$. The word w contains only finitely many positive information $U_i = B((x_i, y_i), r_i)$, $i = 0, ..., n$ on graph(T), i.e.

$\text{graph}(T) \cap U_i \neq \emptyset$ and also only finitely many negative information $U'_i = B((x'_i, y'_i), r'_i) \subseteq \text{graph}(T)^c$ with $i = 0, ..., m$. Since $U'_i \subseteq \text{graph}(T)^c$, we obtain $0 \notin B(y'_i, r'_i)$ for all $i = 0, ..., m$ and since any y'_i is of the form $y'_i = \sum_{j=0}^{k_i} a_{ij} e_j$, the value $j := \max\{k_0, ..., k_m\} + 1$ exists and we obtain $ae_j \notin \bigcup_{i=0}^{m} B(y'_i, r'_i)$ for all $a \in \mathbb{F}$. Correspondingly, any x_i is of the form $x_i = \sum_{j=0}^{l_i} b_{ij} e_j$, the value $\iota := \max\{l_0, ..., l_n\} + 1$ exists and there exists some $\varepsilon > 0$ such that $x_i + \varepsilon e_{\iota'} \in B(x_i, r_i)$ for all $i = 0, ..., n$ and $\iota' \geq \iota$. Now we define a matrix operator $T' : \ell_p \to \ell_p$ which corresponds to the zero matrix except row number j which is

$$\left(1, \underbrace{0, ..., 0}_{\iota-1\text{–times}}, -\frac{b_{00}}{\varepsilon}, -\frac{b_{10}}{\varepsilon}, ..., -\frac{b_{n0}}{\varepsilon}, 0, 0, ...\right).$$

Especially, $T'e_0 = e_j$ and $T'e_{\iota+i} = -\frac{1}{\varepsilon} b_{i0} e_j$ for $i = 0, ..., n$ and $T'e_i = 0$ for all $i \notin \{0, \iota, \iota + 1, ..., \iota + n\}$. Then by choice of j we obtain $U'_i \subseteq \text{graph}(T')^c$ for all $i = 0, ..., m$ and we claim that also $U_i \cap \text{graph}(T') \neq \emptyset$ for all $i = 0, ..., n$: the last statement follows since $x_i + \varepsilon e_{\iota+i} \in B(x_i, r_i)$ for all $i = 0, ..., n$ and $T'(x_i + \varepsilon e_{\iota+i}) = b_{i0} e_j - \varepsilon \frac{1}{\varepsilon} b_{i0} e_j = 0 \in B(y_i, r_i)$. Altogether, w is also a prefix of a name q' of T', i.e. $\delta_{\text{graph}}(q') = T'$ but $T'e_0 = e_j \notin B(0, 1)$ in contrast to the choice of w. Contradiction!

For the remaining results it suffices to consider the case $\ell'_p = \mathcal{B}(\ell_p, \mathbb{F})$. This follows from the fact that $I : \mathcal{B}(\ell_p, \mathbb{F}) \to \mathcal{B}(\ell_p, \ell_p), f \mapsto (x \mapsto (f(x), 0, 0, 0, ...))$ is an isometric embedding such that I, as well as I^{-1}, are (δ, δ)–computable for all representations $\delta \in \{\delta_{\text{graph}}^{<}, \delta_{\text{graph}}, \delta_{\text{seq}}, \delta_{\text{ev}}, \delta_{\text{ev}}^{=}\}$. Hence the case $\mathcal{B}(\ell_p, \ell_p)$ can be reduced to the case $\mathcal{B}(\ell_p, \mathbb{F})$. Analogous considerations yield the results in the case $\mathcal{B}(\ell_p, \mathbb{F}^m)$ with $m > 1$.

"$\delta_{\text{ev}} \not\leq_t \delta_{\text{ev}}^{=}$" Since the mapping $|| \; || :\subseteq \mathcal{C}(\ell_p, \mathbb{F}) \to \mathbb{R}$, defined for linear bounded $T : \ell_p \to \mathbb{F}$ is not continuous by Theorem 13.5, it follows that $\delta_{\text{ev}} \not\leq_t \delta_{\text{ev}}^{=}$. The result for the case $\mathcal{B}(\ell_p, \ell_p)$ follows independently from Theorem 9.7.

"$\delta_{\text{seq}} \not\leq_t \delta_{\text{ev}}$" Let us assume that $\delta_{\text{seq}} \leq_t \delta_{\text{ev}}$. We use the $L :\subseteq \mathbb{F}^{\mathbb{N}} \times \mathbb{R} \to \mathcal{C}(\ell_p, \mathbb{F})$ and the functionals $\lambda_a : \ell_p \to \mathbb{F}$ as in Theorem 13.3. Let q be such that $\frac{1}{p} + \frac{1}{q} = 1$ and let $q = \infty$ if $p = 1$. We consider the function $\lambda :\subseteq \mathbb{R}^{\mathbb{N}} \to \mathcal{B}(\ell_p, \mathbb{F}), a \mapsto \lambda_a$ with $\text{dom}(\lambda) := \mathbb{R}^{\mathbb{N}} \cap \ell_q$. Since $\lambda_a e_i = a_i$ for all $a = (a_n)_{n \in \mathbb{N}} \in \ell_q$, it follows that λ is $(\delta_{\mathbb{R}}^{\mathbb{N}}, \delta_{\text{seq}})$–computable and thus $(\delta_{\mathbb{R}}^{\mathbb{N}}, \delta_{\text{ev}})$–continuous by assumption. Thus, $L^{-1} \circ \lambda :\subseteq \mathbb{R}^{\mathbb{N}} \rightrightarrows \mathbb{F}^{\mathbb{N}} \times \mathbb{R}$ is continuous too and hence the projection $S :\subseteq \mathbb{R}^{\mathbb{N}} \rightrightarrows \mathbb{R}$ on the second component too. Thus S is a continuous operation such that there exists some $s \in S(a)$ for all $a \in \mathbb{R}^{\mathbb{N}}$ with $||a||_q < \infty$ and for all such s the inequality $||a||_q < s$ holds. Such a continuous operation can obviously not exist. Contradiction!

"$\delta_{\text{seq}} \not\leq_t \delta_{\text{graph}}$" This follows from $\delta_{\text{seq}} \not\leq_t \delta_{\text{ev}}$ since $\delta_{\text{graph}} \equiv \delta_{\text{ev}}$ in case of $\mathcal{B}(\ell_p, \mathbb{F})$ by Corollary A.3.

"$\delta^<_{\text{graph}} \not\leq_t \delta_{\text{graph}}$" This follows from $\delta_{\text{seq}} \not\leq_t \delta_{\text{graph}}$ and $\delta_{\text{seq}} \leq \delta^<_{\text{graph}}$. The latter holds by Theorem A.2(1).

"$\delta^<_{\text{graph}} \not\leq_t \delta_{\text{seq}}$" Let us assume that $\delta^<_{\text{graph}} \leq_t \delta_{\text{seq}}$. We consider the function $\lambda :\subseteq \mathbb{R}^{\mathbb{N}} \to \mathcal{B}(\ell_p, \mathbb{F}), a \mapsto \lambda_a$ with $\text{dom}(\lambda) := \mathbb{R}^{\mathbb{N}} \cap \ell_1$ and

$$\lambda_a : \ell_p \to \mathbb{F}, (x_k)_{k \in \mathbb{N}} \mapsto \sum_{k=0}^{\infty} a_k x_k$$

and the function

$$A :\subseteq \mathbb{R}^{\mathbb{N}} \to \mathbb{F}, (a_i)_{i \in \mathbb{N}} \mapsto a_0 2^{\frac{1}{p}} - \sum_{j=0}^{\infty} a_{j+1} 2^{-\frac{j+1}{p}},$$

with $\text{dom}(A) := \mathbb{R}^{\mathbb{N}} \cap \ell_1$. Let $A' :\subseteq \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ be defined by $A'(a_i)_{i \in \mathbb{N}} := (A(a_i)_{i \in \mathbb{N}}, a_1, a_2, ...)$. We claim that $\lambda \circ A' :\subseteq \mathbb{R}^{\mathbb{N}} \to \mathcal{B}(\ell_p, \mathbb{F})$ is $(\delta^{\mathbb{N}}_{\mathbb{R}}, \delta^<_{\text{graph}})$–computable and thus $(\delta^{\mathbb{N}}_{\mathbb{R}}, \delta_{\text{seq}})$–computable by assumption too. But this implies that A is continuous since $A(a) = \lambda_{A'(a)} e_0 = \lambda \circ A'(a)(e_0)$. But obviously A is not continuous, since the value of the sum does substantially depend on coefficients a_j for large j. Contradiction! It remains to prove the claim. Therefore, we define a sequence $e' = (e'_i)_{i \in \mathbb{N}}$ in ℓ_p by $e'_{0j} := 2^{-\frac{j+1}{p}}$ and $e'_{i+1} := e_{i+1}$ for all $i, j \in \mathbb{N}$. Then $||e'_i||_p = 1$ for all $i \in \mathbb{N}$ and $e' : \mathbb{N} \to \ell_p$ is a fundamental sequence, i.e. its linear span is dense in ℓ_p. Now let $a = (a_i)_{i \in \mathbb{N}} \in \ell_1$ be a sequence and let $a' := A'(a)$. Then $\lambda_{a'} e'_i = \lambda_a e_i = a_i$ for all $i \geq 1$ and

$$\lambda_{a'} e'_0 = \sum_{j=0}^{\infty} a'_j e'_{0j} = A(a) 2^{-\frac{1}{p}} + \sum_{j=1}^{\infty} a_j 2^{-\frac{j+1}{p}} = a_0$$

Now given the sequence $a = (a_i)_{i \in \mathbb{N}}$, we can use evaluation and type conversion and the fact that the algebraic operations in ℓ_p are computable to effectively determine the sequence $\alpha_{e'} : \mathbb{N} \to \ell_p$ with $\alpha_{e'} \langle k, \langle n_0, ..., n_k \rangle \rangle := \sum_{i=0}^{k} \alpha_{\mathbb{F}}(n_i) e'_i$. By linearity of $\lambda_{A'(a)}$ it follows

$$\lambda_{A'(a)} \alpha_{e'} \langle k, \langle n_0, ..., n_k \rangle \rangle = \lambda_{A'(a)} \left(\sum_{i=0}^{k} \alpha_{\mathbb{F}}(n_i) e'_i \right) = \sum_{i=0}^{k} \alpha_{\mathbb{F}}(n_i) a_i.$$

Thus, given $(a_i)_{i\in\mathbb{N}}$ we can use type conversion to effectively find the sequence $f:\mathbb{N}\to X\times Y, i\mapsto(\alpha_{e'}(i),\lambda_{A'(a)}\alpha_{e'}(i))$ which is dense in $\mathrm{graph}(\lambda_{A'(a)})$ by continuity of $\lambda_{A'(a)}$ since $\alpha_{e'}$ is dense in X. But this proves that $\lambda\circ A'$ is $(\delta_{\mathbb{R}}^{\mathbb{N}},\delta_{\mathrm{graph}}^{<})$–computable. □

While this theorem shows that the mentioned representations have to be distinguished with respect to computability and continuity, we can deduce from Theorem A.2 (or from Theorem 8.7, alternatively) that the corresponding classes of computable operators coincide. However, the class of computable operators with computable norm is strictly smaller in case of $X=Y=\ell_p$ or $X=\ell_p$ and $Y=\mathbb{F}$ by Corollary 9.5, Example 13.2, respectively.

Corollary A.5 *Let X,Y be computable Banach spaces. Then the subsets of δ–computable operators of $\mathcal{B}(X,Y)$ coincide for $\delta\in\{\delta_{\mathrm{ev}},\delta_{\mathrm{graph}},\delta_{\mathrm{seq}},\delta_{\mathrm{graph}}^{<}\}$. The class of $\delta_{\mathrm{ev}}^{=}$–computable operators is strictly smaller in general.*

In case of the space $\mathcal{B}(\ell_p,\mathbb{F}^m)$ Corollary A.3 leads to a modification of Figure 2 which is displayed in Figure 3. Thus, especially in case of the dual space all considered representations can be ordered linearly w.r.t. computable reducibility.

$$\delta_{\mathrm{ev}}^{=} \longrightarrow \delta_{\mathrm{ev}}\equiv\delta_{\mathrm{graph}} \longrightarrow \delta_{\mathrm{seq}} \longrightarrow \delta_{\mathrm{graph}}^{<}$$

Figure 3: Representations of the operator space $\mathcal{B}(\ell_p,\mathbb{F}^m)$.

In case that both spaces X,Y are finite-dimensional, all considered representations are equivalent, as the following result shows.

Theorem A.6 *Let $n,m\geq 1$ be natural numbers. Then*

$$\delta_{\mathrm{ev}}^{=}\equiv\delta_{\mathrm{ev}}\equiv\delta_{\mathrm{graph}}\equiv\delta_{\mathrm{seq}}\equiv\delta_{\mathrm{graph}}^{<}$$

holds for the corresponding representations of $\mathcal{B}(\mathbb{F}^n,\mathbb{F}^m)$.

Proof. By Theorem A.2 and Corollary A.3 we obtain

$$\delta_{\mathrm{graph}}^{<=}\equiv\delta_{\mathrm{ev}}^{=}\leq\delta_{\mathrm{ev}}\equiv\delta_{\mathrm{graph}}\leq\delta_{\mathrm{seq}}\leq\delta_{\mathrm{graph}}^{<}$$

and thus it suffices to prove $\delta_{\mathrm{graph}}^{<}\leq\delta_{\mathrm{graph}}^{<=}$. In order to prove this it suffices to show that the operator norm $||\ ||:\subseteq\mathcal{C}(\mathbb{F}^n,\mathbb{F}^m)\to\mathbb{R}$ is $(\delta_{\mathrm{graph}}^{<},\delta_{\mathbb{R}})$–computable. Given $\mathrm{graph}(T)\in\mathcal{A}_<(\mathbb{F}^n\times\mathbb{F}^m)$ for some bounded linear operator $T:\mathbb{F}^n\to\mathbb{F}^m$ we can effectively find some function $f:\mathbb{N}\to\mathbb{F}^n\times\mathbb{F}^m$ such that $\mathrm{range}(f)$ is dense in graph(T) by Proposition 4.5. Especially, we can obtain the projections $f_1 : \mathbb{N} \to \mathbb{F}^n$ and $f_2 : \mathbb{N} \to \mathbb{F}^m$ of f. Using the fact that

$$\{(x_1, ..., x_n) \in (\mathbb{F}^n)^n : (x_1, ..., x_n) \text{ linearly independent}\}$$

is an r.e. open subset of $(\mathbb{F}^n)^n$ (cf. [ZB00, BZ00, ZB01]) we can effectively find numbers $i_1, ..., i_n \in \mathbb{N}$ such that $(b_1, ..., b_n) := (f_1(i_1), ..., f_1(i_n))$ is a basis of $\mathbb{F}^n$. Now we can effectively determine the function

$$g : \mathbb{F}^n \to \mathbb{F}^n, (a_1, ..., a_n) \mapsto \sum_{j=1}^{n} a_j b_j$$

by type conversion. Since g is linear and bijective we can effectively determine g^{-1} by the finite-dimensional Inversion Theorem 14.4 and thus we can compute the representation $e_i = \sum_{j=1}^{n} a_{ij} b_j$ of the unit vectors $e_1, ..., e_n \in \mathbb{F}^n$ by $(a_{i1}, ..., a_{in}) := g^{-1}(e_i)$. By linearity of T we obtain

$$y_i := Te_i = T\left(\sum_{j=1}^{n} a_{ij} b_j\right) = \sum_{j=1}^{n} a_{ij} T b_j = \sum_{j=1}^{n} a_{ij} f_2(i_j)$$

for $i = 1, ..., n$ and thus $A := (y_1, ..., y_n) \in \mathbb{F}^{m \times n}$ is the matrix which represents T. Now we obtain

$$||T|| = ||A|| := \max_{i=1,...,m} \sum_{j=1}^{n} |y_{ij}|$$

where $y_j =: (y_{1j}, ..., y_{mj})$ for $j = 1, ..., n$ (see e.g. Example 23.3.b in [Sch97] for the equality $||T|| = ||A||$). Thus, given the graph of T we can actually compute the norm $||T||$ which implies the desired result. □

We close this section with a brief discussion of the inversion operator. On the one hand, it is well-known that inversion $T \mapsto T^{-1}$ is continuous w.r.t. the operator norm topology on the subset of bijective bounded linear operators of $\mathcal{B}(X, Y)$ (cf. Banach's Inversion Stability Theorem 5.6.12 in [Kut96]). On the other hand, we have seen in Section 15 that we cannot admissibly represent $\mathcal{B}(X, Y)$ w.r.t. the operator norm topology in the non-separable case. Moreover, Corollary 6.5 shows that the inversion operator $T \mapsto T^{-1}$ is not continuous w.r.t. δ_{ev} in general. However, Lemma 8.4 shows that the inversion operator is computable w.r.t δ_{graph} (and analogously, one can prove that it is computable w.r.t. $\delta_{\text{graph}}^{<}$). In light of Corollary A.5 this is an interesting observation since it shows that there exist representations of $\mathcal{B}(X, Y)$ which provide both: the ordinary class of computable operators *and* a computable inversion operator.

Corollary A.7 (Inversion) *Let X, Y be computable Banach spaces and consider the inversion mapping*

$$\iota :\subseteq \mathcal{B}(X,Y) \to \mathcal{B}(Y,X), T \mapsto T^{-1}$$

with $\mathrm{dom}(\iota) := \{T \in \mathcal{B}(X,Y) : T \textit{ bijective}\}$. *Then*

(1) ι is $(\delta_{\mathrm{graph}}, \delta_{\mathrm{graph}})$– and $(\delta^{<}_{\mathrm{graph}}, \delta^{<}_{\mathrm{graph}})$–computable,

(2) ι is neither $(\delta_{\mathrm{ev}}, \delta_{\mathrm{ev}})$– nor $(\delta^{=}_{\mathrm{ev}}, \delta^{=}_{\mathrm{ev}})$–continuous in general.

B The Baire Category Theorem

Baire's Category Theorem states that a complete metric space X cannot be decomposed into a countable union of nowhere dense closed subsets A_n (cf. [GP65]). Classically, we can bring this statement into the following two equivalent logical forms:

(1) For all sequences $(A_n)_{n\in\mathbb{N}}$ of closed and nowhere dense subsets $A_n \subseteq X$, there exists some point $x \in X \setminus \bigcup_{n=0}^{\infty} A_n$,

(2) for all sequences $(A_n)_{n\in\mathbb{N}}$ of closed subsets $A_n \subseteq X$ with $X = \bigcup_{n=0}^{\infty} A_n$, there exists some $k \in \mathbb{N}$ such that A_k is somewhere dense.

Both logical forms of the classical theorem have interesting applications. While the first version is often used to ensure the existence of certain types of counterexamples, the second version is for instance used to prove the Open Mapping Theorem and the Closed Graph Theorem [GP65]. However, from the computational point of view the content of both logical forms of the theorem is different. This has already been observed in constructive analysis, where a discussion of the theorem can be found in [BR87]. We will study the theorem from the point of view of computable analysis. In this spirit one version of the Baire Category Theorem has already been proved by Yasugi, Mori and Tsujii [YMT99].

Depending on how the sequence $(A_n)_{n\in\mathbb{N}}$ is represented, i.e. how it is "given", we can compute an appropriate point x in case of the first version or compute a suitable index k in case of the second version. Roughly speaking, the second logical version requires stronger information on the sequence of sets than the first version. Unfortunately, this makes the second version of the theorem less applicable than its classical counterpart, since this strong type of information on the sequence $(A_n)_{n\in\mathbb{N}}$ is rarely available.

B.1 First Computable Baire Category Theorem

For this section let (X, d, α) be some fixed complete computable metric space, let $\mathcal{C}(X) := \mathcal{C}(X, \mathbb{R})$ and let $\mathcal{A} := \mathcal{A}(X)$ be the set of closed subsets of X. We write $\mathcal{A}_>$ to indicate that we use the represented space $(\mathcal{A}, \delta_{\mathcal{A}}^>)$. In the following we will use the fact that the union operation is computable on $\mathcal{A}_>$.

Proposition B.1 *The operation $\mathcal{A}_> \times \mathcal{A}_> \to \mathcal{A}_>, (A, B) \mapsto A \cup B$ is computable.*

Proof. Using evaluation and type conversion w.r.t. $[\delta_X \to \delta_{\mathbb{R}}]$, it is straightforward to show that $\mathcal{C}(X) \times \mathcal{C}(X) \to \mathcal{C}(X), (f, g) \mapsto f \cdot g$ is computable, but if $f^{-1}\{0\} = A$ and $g^{-1}\{0\} = B$, then $(f \cdot g)^{-1}\{0\} = A \cup B$. Thus the desired result follows from the previous Proposition 4.4. □

Since computable functions have the property that they map computable points to computable points, we can deduce that the class of co-r.e. closed sets is closed under intersection.

Corollary B.2 *If $A, B \subseteq X$ are co-r.e. closed, then $A \cup B$ is co-r.e. closed too.*

Moreover, it is obvious that we can compute complements of open balls in the following sense.

Proposition B.3 *$(X \setminus B(\alpha(n), \overline{k}))_{\langle n,k \rangle \in \mathbb{N}}$ is a computable sequence in $\mathcal{A}_>$.*

Using these both observations, we can prove the following first version of the computable Baire Category Theorem just by transferring the classical proof.

Theorem B.4 (First computable Baire Category Theorem) *There exists a computable operation $\Delta :\subseteq \mathcal{A}_>^{\mathbb{N}} \rightrightarrows X^{\mathbb{N}}$ with the following property: for any sequence $(A_n)_{n \in \mathbb{N}}$ of closed nowhere dense subsets of X, there exists some sequence $(x_n)_{n \in \mathbb{N}} \in \Delta(A_n)_{n \in \mathbb{N}}$ and all such sequences $(x_n)_{n \in \mathbb{N}}$ are dense in $X \setminus \bigcup_{n=0}^{\infty} A_n$.*

Proof. Let us fix some $n = \langle n_1, n_2 \rangle \in \mathbb{N}$. We construct sequences $(x_{n,k})_{k \in \mathbb{N}}$ in X and $(r_{n,k})_{k \in \mathbb{N}}$ in $\mathbb{Q}$ as follows: let $x_{\langle n_1, n_2 \rangle, 0} := \alpha(n_1)$, $r_{\langle n_1, n_2 \rangle, 0} := 2^{-n_2}$. Given $r_{n,i}$ and $x_{n,i}$ we can effectively find some point $x_{n,i+1} \in \text{range}(\alpha) \subseteq X$ and a rational $\varepsilon_{n,i+1}$ with $0 < \varepsilon_{n,i+1} \leq r_{n,i}$ such that

$$B(x_{n,i+1}, \varepsilon_{n,i+1}) \subseteq (X \setminus A_i) \cap B(x_{n,i}, r_{n,i}) = (A_i \cup X \setminus B(x_{n,i}, r_{n,i}))^c.$$

One the one hand, such a point and radius have to exist since A_i is nowhere dense and on the other hand, we can effectively find them, given a $\delta_{\mathcal{A}}^{>\mathbb{N}}$–name of the sequence $(A_n)_{n\in\mathbb{N}}$ and using Propositions B.1 and B.3. Now let $r_{n,i+1} := \varepsilon_{n,i+1}/2$. Altogether, we obtain a sequence of closed balls

$$\overline{B}(x_{n,i+1}, r_{n,i+1}) \subseteq \overline{B}(x_{n,i}, r_{n,i}) \subseteq ... \subseteq \overline{B}(x_{n,0}, r_{n,0})$$

with $r_{n,i} \leq 2^{-i}$ and thus $x_n := \lim_{i\to\infty} x_{n,i}$ exists since X is complete and the sequence $(x_{n,i})_{i\in\mathbb{N}}$ is even rapidly converging. Finally, the sequence $(x_n)_{n\in\mathbb{N}}$ is dense in $X \setminus \bigcup_{n=0}^{\infty} A_n$, since for any pair (n_1, n_2) we obtain by definition $x_{\langle n_1,n_2\rangle} \in B(\alpha(n_1), 2^{-n_2})$. Altogether, the construction shows how a Turing machine can transform each $\delta_{\mathcal{A}}^{>\mathbb{N}}$–name of a sequence $(A_n)_{n\in\mathbb{N}}$ into a δ_X–name of a suitable sequence $(x_n)_{n\in\mathbb{N}}$. $\square$

As a direct corollary of this uniformly computable version of the Baire Category Theorem we can conclude the following weak version.

Corollary B.5 *For any computable sequence $(A_n)_{n\in\mathbb{N}}$ of co-r.e. closed nowhere dense subsets $A_n \subseteq X$, there exists some computable sequence $(x_n)_{n\in\mathbb{N}}$ which is dense in $X \setminus \bigcup_{n=0}^{\infty} A_n$.*

Since any computable sequence $(A_n)_{n\in\mathbb{N}}$ of co-r.e. closed nowhere dense subsets $A_n \subseteq X$ is "sequentially effectively nowhere dense" in the sense of Yasugi, Mori and Tsujii, we can conclude the previous corollary also from their effective Baire Category Theorem [YMT99].

It is a well-known fact that the set of computable real numbers $\mathbb{R}_c$ cannot be enumerated by a computable sequence [Wei00]. We obtain a new proof for this fact and a generalization for computable complete metric spaces without isolated points. First we prove the following simple proposition.

Proposition B.6 *The operation $X \to \mathcal{A}_>, x \mapsto \{x\}$ is computable.*

Proof. This follows directly from the fact that $d : X \times X \to \mathbb{R}$ is computable and $\{x\} = X \setminus \bigcup\{B(\alpha(n), \overline{k}) : d(\alpha(n), x) > \overline{k}$ and $n, k \in \mathbb{N}\}$. $\square$

If X is a metric space without isolated points, then all singleton sets $\{x\}$ are nowhere dense closed subsets. This allows to combine the previous proposition with the computable Baire Category Theorem B.5.

Corollary B.7 *If X is a computable complete metric space without isolated points, then for any computable sequence $(y_n)_{n\in\mathbb{N}}$ in X, there exists a computable sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $(x_n)_{n\in\mathbb{N}}$ is dense in $X \setminus \{y_n : n \in \mathbb{N}\}$.*

Using Theorem B.4 it is straightforward to derive even a uniform version of this theorem which states that we can effectively find a corresponding sequence $(x_n)_{n\in\mathbb{N}}$ for any given sequence $(y_n)_{n\in\mathbb{N}}$. Instead of formulating this uniform version, we include the following corollary which generalizes the statement that $\mathbb{R}_c$ cannot be enumerated by a computable sequence.

Corollary B.8 *If X is a computable complete metric space without isolated points, then there exists no computable sequence $(y_n)_{n\in\mathbb{N}}$ such that $\{y_n : n \in \mathbb{N}\}$ is the set of computable points of X.*

B.2 Computable but Nowhere Differentiable Functions

In this section we want to effectivize the standard example of an application of the Baire Category Theorem. We will show that there exists a computable but nowhere differentiable function $f : [0,1] \to \mathbb{R}$. It is not to difficult to construct an example of such a function directly and actually, some typical examples of continuous nowhere differentiable functions, like *van der Waerden's function* $f : [0,1] \to \mathbb{R}$ or *Riemann's function* $g : [0,1] \to \mathbb{R}$ (cf. [Kut96]), defined by

$$f(x) := \sum_{n=0}^{\infty} \frac{\langle 4^n x\rangle}{4^n} \quad \text{and} \quad g(x) := \sum_{n=0}^{\infty} \frac{\sin(n^2\pi x)}{n^2},$$

where $\langle x\rangle := \min\{x - [x], 1 + [x] - x\}$ denotes the distance of x to the nearest integer, can easily be seen to be computable. The purpose of this section is rather to demonstrate that the computable version of the Baire Category Theorem can be applied in similar situations as the classical one.

In this section we will use the computable Banach space of continuous functions $(\mathcal{C}[0,1], || \, ||, e)$ with $\mathcal{C}[0,1] := \mathcal{C}([0,1],\mathbb{R})$, as defined in Proposition 3.8(4). By $\delta_{\mathcal{C}}$ we denote the Cauchy representation of this space and in the following we tacitly assume that $\mathcal{C}[0,1]$ is endowed with this representation. For technical simplicity we assume that functions $f : [0,1] \to \mathbb{R}$ are actually functions $f : \mathbb{R} \to \mathbb{R}$ extended constantly, i.e. $f(x) = f(0)$ for $x \leq 0$ and $f(x) = f(1)$ for $x \geq 1$. It is well-known that a function $f : [0,1] \to \mathbb{R}$ is $\delta_{\mathcal{C}}$–computable, if it is computable considered as a function $f : \mathbb{R} \to \mathbb{R}$ and we can actually replace $\delta_{\mathcal{C}}$ by the restriction of $[\delta_{\mathbb{R}} \to \delta_{\mathbb{R}}]$ to $\mathcal{C}[0,1]$ whenever it is helpful [Wei00].

We will consider differentiability for functions $f : [0,1] \to \mathbb{R}$ only within $[0,1]$. If a function $f : [0,1] \to \mathbb{R}$ is differentiable at some point $t \in [0,1]$, then the quotient $|\frac{f(t+h)-f(t)}{h}|$ is bounded for all $h \neq 0$. Thus f belongs to the set

$$D_n := \left\{ f \in \mathcal{C}[0,1] : (\exists t \in [0,1])(\forall h \in \mathbb{R} \setminus \{0\}) \left| \frac{f(t+h) - f(t)}{h} \right| \leq n \right\}$$

for some $n \in \mathbb{N}$. Because of continuity of the functions f, it suffices if the universal quantification over h ranges over some dense subset of $\mathbb{R} \setminus \{0\}$ such as $\mathbb{Q} + \pi$ in order to obtain the same set D_n.

It is well-known, that all sets D_n are closed and nowhere dense [GP65]. Thus, by the classical Baire Category Theorem, the set $\mathcal{C}[0,1] \setminus \bigcup_{n=0}^{\infty} D_n$ is non-empty and there exists some continuous but nowhere differentiable function $f : [0,1] \to \mathbb{R}$. Our aim is to prove that $(D_n)_{n \in \mathbb{N}}$ is a computable sequence of co-r.e. closed nowhere dense subsets of $\mathcal{C}[0,1]$, i.e. a computable sequence in $\mathcal{A}_>(\mathcal{C}[0,1])$. Then we can apply the computable Baire Category Theorem B.4 to ensure the existence of a computable but nowhere differentiable function $f : [0,1] \to \mathbb{R}$.

The crucial point is to get rid of the existential quantification of t over $[0,1]$ since arbitrary unions of co-r.e. closed sets need not to be (co-r.e.) closed again. The main tool will be the following Proposition which roughly speaking states that co-r.e. closed subsets are closed under parametrized countable and computable intersection and compact computable union.

Proposition B.9 *Let (X, δ) be some represented space and let (Y, d, α) be some computable metric space.*

(1) If the function $A : X \times \mathbb{N} \to \mathcal{A}_>(Y)$ is computable, then the countable intersection $\cap A : X \to \mathcal{A}_>(Y), x \mapsto \bigcap_{n=0}^{\infty} A(x,n)$ is computable too.

(2) If the function $U : X \times \mathbb{R} \to \mathcal{A}_>(Y)$ is computable, then the compact union $\cup U : X \to \mathcal{A}_>(Y), x \mapsto \bigcup_{t \in [0,1]} U(x,t)$ is computable too.

Proof. (1) Let $A : X \times \mathbb{N} \to \mathcal{A}_>(Y)$ be computable. If for some fixed $x \in X$ we have $A(x,n) = Y \setminus \bigcup_{k=0}^{\infty} B(\alpha(i_{nk}), \overline{j_{nk}})$ with $i_{nk}, j_{nk} \in \mathbb{N}$ for all $n, k \in \mathbb{N}$, then

$$\bigcap_{n=0}^{\infty} A(x,n) = \bigcap_{n=0}^{\infty} \left(Y \setminus \bigcup_{k=0}^{\infty} B(\alpha(i_{nk}), \overline{j_{nk}}) \right) = Y \setminus \left(\bigcup_{\langle n,k \rangle = 0}^{\infty} B(\alpha(i_{nk}), \overline{j_{nk}}) \right).$$

Thus, it is straightforward to show that $\cap A : X \to \mathcal{A}_>(Y)$ is computable too.

(2) Now let $U : X \times \mathbb{R} \to \mathcal{A}_>(Y)$ be computable. Let $\delta_{[0,1]} :\subseteq \Sigma^\omega \to [0,1]$ be the *signed digit representation* of the unit interval, where $\Sigma = \{0, 1, -1\}$ and $\delta_{[0,1]}$ is defined in all possible cases by

$$\delta_{[0,1]}(p) := \sum_{i=0}^{\infty} p(i) 2^{-i}.$$

It is known that $\mathrm{dom}(\delta_{[0,1]})$ is compact and $\delta_{[0,1]}$ is computably equivalent to the Cauchy representation $\delta_{\mathbb{R}}$, restricted to $[0,1]$ (cf. [Wei00]). Thus, U restricted to $X \times [0,1]$ is $([\delta,\delta_{[0,1]}],\delta_{\mathcal{A}}^{>})$–computable. Then there exists some Turing machine M which computes a function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ which is a $([\delta,\delta_{[0,1]}],\delta_{\mathcal{A}}^{>})$–realization of $U : X \times \mathbb{R} \to \mathcal{A}_{>}(Y)$. Thus, for each given input sequence $\langle p,q\rangle \in \Sigma^\omega$ with $x := \delta(p)$ and $t := \delta_{[0,1]}(q)$ the machine M produces some output sequence $01^{\langle n_{q0},k_{q0}\rangle}01^{\langle n_{q1},k_{q1}\rangle}01^{\langle n_{q2},k_{q2}\rangle}\ldots$ such that

$$U(x,t) = Y \setminus \bigcup_{i=0}^{\infty} B(\alpha(n_{qi}),\overline{k_{qi}}).$$

Since we will only consider a fixed p, we do not mention the corresponding dependence in the indices of the values n_{qi}, k_{qi}. It is easy to prove that the set $W := \{w \in \Sigma^* : (\exists q \in \mathrm{dom}(\delta_{[0,1]}))\ w \text{ is a prefix of } q\}$ is recursive.

We will sketch the construction of a machine M' which computes the operation $\cup U : X \to \mathcal{A}_{>}(Y)$. On input p the machine M' works in parallel phases $\langle i,j,k\rangle = 0,1,2,\ldots$ and produces an output r. In phase $\langle i,j,k\rangle$ it simulates M on input $\langle p, w0^\omega\rangle$ for all words $w \in \Sigma^k \cap W$ and exactly k steps. Let $01^{\langle n_{w0},k_{w0}\rangle}01^{\langle n_{w1},k_{w1}\rangle}\ldots01^{\langle n_{wl_w},k_{wl_w}\rangle}0$ be the corresponding output of M (more precisely: the longest prefix of the output which ends with 0). Then the machine M' checks whether for all $w \in \Sigma^k \cap W$ there is some $\iota_w = 0,\ldots,l_w$ such that $d(\alpha(i),\alpha(n_{w\iota_w})) + \overline{j} < \overline{k_{w\iota_w}}$ holds, which especially implies

$$B(\alpha(i),\overline{j}) \subseteq \bigcap_{w\in\Sigma^k\cap W} B(\alpha(n_{w\iota_w}),\overline{k_{w\iota_w}}) \subseteq \bigcap_{t\in[0,1]} Y \setminus U(x,t) = Y \setminus \bigcup U(x).$$

The verification is possible since (X,d,α) is a computable metric space. As soon as corresponding values ι_w are found for all $w \in \Sigma^k \cap W$, phase $\langle i,j,k\rangle$ is finished with extending the output by $01^{\langle i,j\rangle}$. Otherwise it might happen that the phase never stops, but other phases may run in parallel.

We claim that this machine M' actually computes $\cup U$. On the one hand, it is clear that $B(\alpha(i),\overline{j}) \subseteq Y \setminus \cup U(x)$ whenever $01^{\langle i,j\rangle}$ is written on the output tape by M'. Thus, if M' actually produces an infinite output r, then we obtain immediately $\delta_{\mathcal{A}}^{>}(r) \subseteq \cup U(\delta(p))$. On the other hand, let $y \in Y \setminus \cup U(\delta(p))$. Then for any $q \in \mathrm{dom}(\delta_{[0,1]})$ the machine M produces some output sequence $01^{\langle n_{q0},k_{q0}\rangle}01^{\langle n_{q1},k_{q1}\rangle}01^{\langle n_{q2},k_{q2}\rangle}\ldots$ and there has to be some l_q such that $y \in B(\alpha(n_{ql_q}),\overline{k_{ql_q}})$ and a finite number k of steps such that M produces $01^{\langle n_{ql_q},k_{ql_q}\rangle}0$ on the output tape. Since $\mathrm{dom}(\delta_{[0,1]})$ is compact, there is even a common such k for all $q \in \mathrm{dom}(\delta_{[0,1]})$. Let $w' := w0^\omega$ for all $w \in \Sigma^*$. Then there exist $i,j \in \mathbb{N}$ such that

$$y \in B(\alpha(i),\overline{j}) \subseteq \bigcap_{w\in\Sigma^k\cap W} B(\alpha(n_{w'l_{w'}}),\overline{k_{w'l_{w'}}})$$

and $d(\alpha(i), \alpha(n_{w'l_{w'}})) + \overline{j} < \overline{k_{w'l_{w'}}}$. Thus M' will produce $01^{\langle i,j \rangle}$ on the output tape in phase $\langle i,j,k \rangle$. Altogether, this proves $\delta_{\mathcal{A}}^{>}(r) = \cup U(\delta(p))$ and thus $\cup U : X \to \mathcal{A}_{>}(Y)$ is computable. □ □

Now using this proposition, we can directly prove the desired result.

Theorem B.10 *There exists a computable sequence $(f_n)_{n \in \mathbb{N}}$ of computable but nowhere differentiable functions $f_n : [0,1] \to \mathbb{R}$ such that $\{f_n : n \in \mathbb{N}\}$ is dense in $\mathcal{C}[0,1]$.*

Proof. If we can prove that $(D_n)_{n \in \mathbb{N}}$ is a computable sequence of co-r.e. nowhere dense closed sets, then Corollary B.5 implies the existence of a computable sequence of computable functions f_n in $\mathcal{C}[0,1] \setminus \bigcup_{n=0}^{\infty} D_n$. Since all somewhere differentiable functions are included in some D_n, it follows that all f_n are nowhere differentiable. Since it is well-known that all D_n are nowhere dense, it suffices to prove the computability property. We recall that it suffice to consider values $h \in \mathbb{Q} + \pi$ in the definition of D_n because of continuity of the functions f. We define a function $F : \mathbb{N} \times \mathbb{R} \times \mathbb{N} \times \mathcal{C}[0,1] \to \mathbb{R}$ by

$$F(n,t,k,f) := \max\left\{\left|\frac{f(t+\overline{k}+\pi) - f(t)}{\overline{k}+\pi}\right| - n, 0\right\}.$$

Then using the evaluation property of $[\delta_{\mathbb{R}} \to \delta_{\mathbb{R}}]$, one can prove that F is computable. Using type conversion w.r.t. $[\delta_{\mathcal{C}} \to \delta_{\mathbb{R}}]$ one obtains computability of $\hat{F} : \mathbb{N} \times \mathbb{R} \times \mathbb{N} \to \mathcal{C}(\mathcal{C}[0,1])$, defined by $\hat{F}(n,t,k)(f) := F(n,t,k,f)$. Using Proposition 4.4 we can conclude that the mapping $A : \mathbb{N} \times \mathbb{R} \times \mathbb{N} \to \mathcal{A}_{>}(\mathcal{C}[0,1])$ with $A(n,t,k) := (\hat{F}(n,t,k))^{-1}\{0\}$ is computable. Thus by the previous proposition $\cap A : \mathbb{N} \times \mathbb{R} \to \mathcal{A}_{>}(\mathcal{C}[0,1])$ is also computable and thus $\cup \cap A : \mathbb{N} \to \mathcal{A}_{>}(\mathcal{C}[0,1])$ too. Now we obtain

$$\cup \cap A(n) = \bigcup_{t \in [0,1]} \bigcap_{k=0}^{\infty} \left\{ f \in \mathcal{C}[0,1] : \left|\frac{f(t+\overline{k}+\pi) - f(t)}{\overline{k}+\pi}\right| \leq n \right\} = D_n.$$

Thus, $(D_n)_{n \in \mathbb{N}}$ is a computable sequence of co-r.e. closed subsets of $\mathcal{C}[0,1]$. □

B.3 Second Computable Baire Category Theorem

While the first version of the computable Baire Category Theorem has been proved by a direct adaptation of the classical proof, the second version will even be a consequence of the classical version. Whenever a classical theorem for complete computable metric spaces X, Y has the form

$$(\forall x)(\exists y) R(x,y)$$

with a predicate $R \subseteq X \times Y$ which can be proven to be r.e. open, then the theorem admits a computable multi-valued realization $F : X \rightrightarrows Y$ such that $R(x, y)$ holds for all $y \in F(x)$ (cf. the Uniformization Theorem 3.2.40 in [Bra99b]). Actually, a computable version of the second formulation of the Baire Category Theorem, given in the Introduction, can be derived as such a direct corollary of the classical version.

Given a co-r.e. set $A \subseteq X$, the closure of its complement $\overline{A^c}$ needs not to be co-r.e. again (cf. Proposition 5.4 in [Bra99a]). Thus, the "complement closure" operation $\mathcal{A}_> \to \mathcal{A}_>, A \mapsto \overline{A^c}$ cannot be computable (and actually it is not even continuous in the corresponding way). In order to overcome this deficiency, we can simply include the information on $\overline{A^c}$ into a representing sequence of A. This is a usual trick in topology and computable analysis to make functions continuous or computable, respectively. So, if δ is an arbitrary representation of $\mathcal{A}$, then the representation δ^+ of $\mathcal{A}$, defined by

$$\delta^+\langle p, q\rangle := A :\iff \delta(p) = A \text{ and } \delta^>_{\mathcal{A}}(q) = \overline{A^c},$$

has automatically the property that $\mathcal{A} \to \mathcal{A}, A \mapsto \overline{A^c}$ becomes $(\delta^+, \delta^>_{\mathcal{A}})$–computable. Here $\langle\ \rangle : \Sigma^\omega \times \Sigma^\omega \to \Sigma^\omega$ denotes some appropriate computable pairing function [Wei00]. We can especially apply this procedure to $\delta := \delta^>_{\mathcal{A}}$. The corresponding $\delta^{>+}_{\mathcal{A}}$–computable sets $A \subseteq X$ are called *bi-co-r.e.* closed sets. In this case we write $\mathcal{A}_{>+}$ to denote the represented space $(\mathcal{A}, \delta^{>+}_{\mathcal{A}})$. Now we can directly conclude that the property "somewhere dense" is r.e.

Proposition B.11 *The set $\{A \in \mathcal{A} : A$ is somewhere dense$\}$ is r.e. in $\mathcal{A}_{>+}$.*

The proof follows directly from the fact that a closed set $A \subseteq X$ is somewhere dense, if and only if there exist $n, k \in \mathbb{N}$ such that $B(\alpha(n), \overline{k}) \subseteq A^\circ = \overline{A^c}^{\,c}$. We can now directly conclude the second computable version of the Baire Category Theorem as a consequence of the classical version (and thus especially as a consequence of the first computable Baire Category Theorem B.4).

Theorem B.12 (Second computable Baire Category Theorem) *There exists a computable operation $\Sigma :\subseteq \mathcal{A}^{\mathbb{N}}_{>+} \rightrightarrows \mathbb{N}$ with the following property: for any sequence $(A_n)_{n\in\mathbb{N}}$ of closed subsets of X with $X = \bigcup_{n=0}^\infty A_n$, there exists some $\langle i, j, k\rangle \in \Sigma(A_n)_{n\in\mathbb{N}}$ and for all such $\langle i, j, k\rangle$ we obtain $B(\alpha(i), \overline{j}) \subseteq A_k$.*

Of course, if we replace $\mathcal{A}_{>+}$ by $(\mathcal{A}, \delta^+)$ with any other underlying representation δ instead of $\delta^>_{\mathcal{A}}$, then the theorem would also hold true. We mention that the corresponding constructive version of the theorem (Theorem 2.5 in [BR87]), if directly translated into a computable version, leads to a weaker statement than Theorem B.12: if the sequence $(A_n)_{n\in\mathbb{N}}$ would be effectively given by the sequences of distance functions of A and A^c, this would constitute a stronger input information than it is the case if it is given by $\delta_{\mathcal{A}}^{>+}$. Now we can formulate a weak version of the second Baire Category Theorem.

Corollary B.13 *For any computable sequence $(A_n)_{n\in\mathbb{N}}$ of bi-co-r.e. closed subsets $A_n \subseteq X$ with $X = \bigcup_{j=0}^{\infty} A_{\langle i,j\rangle}$ for all $i \in \mathbb{N}$, there exists a total computable function $f : \mathbb{N} \to \mathbb{N}$ such that $A_{\langle i,f(i)\rangle}$ is somewhere dense for all $i \in \mathbb{N}$.*

By applying some techniques from recursion theory [Wei87, Odi89], we can prove that the previous theorem and its corollary do not hold true with $\mathcal{A}_>$ or $\mathcal{A}$ instead of $\mathcal{A}_{>+}$. For this result we use as metric space the Euclidean space $X = \mathbb{R}$.

Theorem B.14 *There exists a computable sequence $(A_n)_{n\in\mathbb{N}}$ of recursive closed subsets $A_n \subseteq [0,1]$ with $[0,1] = \bigcup_{j=0}^{\infty} A_{\langle i,j\rangle}$ for all $i \in \mathbb{N}$ such that for every computable $f : \mathbb{N} \to \mathbb{N}$ there is some $i \in \mathbb{N}$ such that $A_{\langle i,f(i)\rangle}$ is nowhere dense.*

Proof. We use some total Gödel numbering $\varphi : \mathbb{N} \to P$ of the set of partial recursive functions $P := \{f :\subseteq \mathbb{N} \to \mathbb{N} : f \text{ computable}\}$ to define sets

$$A'_{\langle i,j\rangle} := \overline{\bigcup_{k=0}^{\min \varphi_i^{-1}\{j\}} \left\{\frac{m}{2^k} : m = 0, ..., 2^k\right\}}.$$

For this definition we assume $\min \emptyset = \infty$. Whenever $i \in \mathbb{N}$ is the index of some total recursive function $\varphi_i : \mathbb{N} \to \mathbb{N}$ such that $\operatorname{range}(\varphi_i) \neq \mathbb{N}$, then we obtain $\bigcup_{j=0}^{\infty} A'_{\langle i,j\rangle} = [0,1]$ and $A'_{\langle i,j\rangle}$ is somewhere dense, if and only if $j \notin \operatorname{range}(\varphi_i)$. Using the smn-Theorem one can inductively prove that there is a total recursive function $r : \mathbb{N} \to \mathbb{N}$ such that $\varphi_{r\langle i,j\rangle}$ is total if φ_i is and

$$\operatorname{range}(\varphi_{r\langle i,\langle k,\langle n_0,...,n_k\rangle\rangle\rangle}) = \operatorname{range}(\varphi_i) \cup \{n_0, ..., n_k\}.$$

Let i_0 be the index of some total recursive function which enumerates some simple set $S := \operatorname{range}(\varphi_{i_0})$ and define $A_{\langle i,j\rangle} := A'_{\langle r\langle i_0,i\rangle,j\rangle}$. Then $(A_n)_{n\in\mathbb{N}}$ is a computable sequence of recursive closed subsets $A_n \subseteq [0,1]$. Let us assume that there exists a total recursive function $f : \mathbb{N} \to \mathbb{N}$ with the property that $A_{\langle i,f(i)\rangle}$ is somewhere dense for all $i \in \mathbb{N}$. Let $j_0 \in \mathbb{N} \setminus S$ and define a function $g : \mathbb{N} \to \mathbb{N}$ inductively by $g(0) := j_0$ and $g(n+1) := f(r\langle i_0, \langle n, \langle g(0), ...g(n)\rangle\rangle\rangle)$. By induction we prove $g(n) \in \mathbb{N} \setminus (S \cup \{g(0), ..., g(n-1)\})$ for all $n \in \mathbb{N}$. The case $n = 0$ is obvious since $g(0) = j_0 \in \mathbb{N} \setminus S$. Now, let us assume that $g(0), ..., g(n) \in \mathbb{N} \setminus S$ has been proved and let $i := r\langle i_0, \langle n, \langle g(0), ..., g(n)\rangle\rangle\rangle$. By assumption $A_{\langle i,f(i)\rangle}$ is somewhere dense, hence

$$g(n+1) = f(i) \notin \operatorname{range}(\varphi_i) = \operatorname{range}(\varphi_{i_0}) \cup \{g(0), ..., g(n)\} = S \cup \{g(0), ..., g(n)\}.$$

Thus, g is computable and range(g) is some infinite r.e. subset of the immune set $\mathbb{N} \setminus S$. Contradiction! □

Even a simpler variant of the same idea can be used to prove that in a well-defined sense there exists no continuous multi-valued operation $\Sigma :\subseteq \mathcal{A}_>^{\mathbb{N}} \rightrightarrows \mathbb{N}$ which meets the conditions of Theorem B.12.

Unfortunately, the simplicity of the proof of the second computable Baire Category Theorem B.12 corresponds to its uselessness. The type of information that one could hope to gain from an application of the theorem has already to be fed in by the input information. However, Theorem B.14 shows that a substantial improvement of Theorem B.12 seems to be impossible.

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