Computing the Dimension of Linear Subspaces

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Abstract. Since its very beginning, linear algebra is a highly algorithmic subject. Let us just mention the famous Gauß Algorithm which was invented before the theory of algorithms has been developed. The purpose of this paper is to link linear algebra explicitly to computable analysis, that is the theory of computable real number functions. Especially, we will investigate in which sense the dimension of a given linear subspace can be computed. The answer highly depends on how the linear subspace is given: if it is given by a finite number of vectors whose linear span represents the space, then the dimension does not depend continuously on these vectors and consequently it cannot be computed. If the linear subspace is represented via its distance function, which is a standard way to represent closed subspaces in computable analysis, then the dimension does computably depend on the distance function.

1 Introduction

Computational aspects of linear algebra are mostly studied with respect to their algebraic complexity, that is, in machine models capable of processing in unit time real numbers. Digital computers can however work on finite information like integers or floating point numbers as approximation of reals. The common implicit believe (or hope) is that, as precision increases, a program's approximate output tends to the desired exact result.

We investigate computational aspects of linear algebra from the somewhat different point of view of computable analysis, which offers a precise framework for treating computability aspects of real number computations based on Turing machines. Starting with Turing's own famous paper [9] real number computations have been investigated using his machine model. Later on, the theory has been further developed by Grzegorczyk [4], Lacombe [6], Pour-El and Richards [8], Kreitz and Weihrauch [10], Ko [5] and many others. We have essentially adopted Weihrauch's approach [10], the so-called *Type-2-Theory of Effectivity*, which allows to express computations with real number, continuous functions and subsets in a highly uniform way.

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In constructive analysis [1], as well as in computable analysis closed subsets are often represented by distance functions, which, roughly speaking, play the role of continuous substitutes for characteristic functions (cf. [2] for a survey). Such representations of sets by distance functions can also be considered for constructions of data structures for solid modeling and other CAD applications (cf. [7]). If it appears that the result of a computation with sets is a linear subspace, one is interested in computing the dimension and a basis of this space. Our main result proves that both is possible.

The following section contains a short introduction to computable analysis and the notions used there. Section 3 presents our computability results on the dimension of linear subspaces.

2 Computable Analysis

In this section we briefly present some basic notions from computable analysis (based on the approach of Type-2 theory of effectivity) and some direct consequences of well-known facts. For a precise and comprehensive reference we refer the reader to [10]. Roughly speaking, a partial real number function $f :\subseteq \mathbb{R}^n \to \mathbb{R}$ is computable, if there exists a Turing machine which transfers each sequence $p \in \Sigma^\omega$ that represents some input $x \in \mathbb{R}^n$ into some sequence $F_M(p)$ which represents the output $f(x)$. Since the set of real numbers has continuum cardinality, real numbers can only be represented by infinite sequences $p \in \Sigma^\omega$ (over some finite alphabet Σ) and thus, such a Turing machine M has to compute infinitely long. But in the long run it transfers each input sequence p into an appropriate output sequence $F_M(p)$. It is reasonable to allow only one-way output tapes for infinite computations since otherwise the output after finite time would be useless (because it could possibly be replaced later by the machine). It is straightforward how this notion of computability can be generalized to other sets X with a corresponding *representation*, that is a surjective partial mapping $\delta :\subseteq \Sigma^\omega \to X$.

Definition 1 (Computable functions). Let δ, δ' be representations of X, Y, respectively. A function $f :\subseteq X \to Y$ is called (δ, δ')*–computable*, if there exists some Turing machine M such that $\delta' F_M(p) = f\delta(p)$ for all $p \in \mathrm{dom}(f\delta)$.

Here, $F_M :\subseteq \Sigma^\omega \to \Sigma^\omega$ denotes the partial function, computed by the Turing machine M. It is straightforward how to generalize this definition to functions with several inputs and it can even be generalized to multi-valued operations $f :\subseteq X \rightrightarrows Y$, where $f(x)$ is a subset of Y instead of a single value. In this case we replace the condition in the definition above by $\delta' F_M(p) \in f\delta(p)$.

Already in case of the real numbers it appears that the defined notion of computability sensitively relies on the chosen representation of the real numbers. The theory of *admissible* representations completely answers the question how to find "reasonable" representations of topological spaces [10]. Let us just mention that for admissible representations δ, δ' each (δ, δ')–computable function is necessarily continuous (w.r.t. the final topologies of δ, δ').

An example of an admissible representation of the real numbers is the so-called *Cauchy representation* $\rho :\subseteq \Sigma^\omega \to \mathbb{R}$, where roughly speaking, $\rho(p) = x$ if p is an (appropriately encoded) sequence of rational numbers $(q_i)_{i\in\mathbb{N}}$ which converges rapidly to x, i.e. $|x_i - x_k| \leq 2^{-k}$ for all $i > k$. By standard coding techniques this representation can easily be generalized to a representation of the n-dimensional Euclidean space $\rho^n :\subseteq \Sigma^\omega \to \mathbb{R}^n$ and to a representation of $m \times n$ matrices $\rho^{m\times n} :\subseteq \Sigma^\omega \to \mathbb{R}^{m\times n}$.

If δ, δ' are admissible representations of topological spaces X, Y, respectively, then there exists a canonical representation $[\delta, \delta'] :\subseteq \Sigma^\omega \to X \times Y$ of the product $X \times Y$ and a canonical representation $[\delta \to \delta'] :\subseteq \Sigma^\omega \to C(X, Y)$ of the space $C(X, Y)$ of the total continuous functions $f : X \to Y$. We just mention that these representations allow evaluation and type conversion (which correspond to an utm- and smn-Theorem). Evaluation means that the evaluation function $C(X, Y)\times X \to Y, (f, x) \mapsto f(x)$ is $([[\delta \to \delta'], \delta], \delta')$–computable and type conversion means that a function $f : Z \times X \to Y$ is $([\delta'', \delta], \delta')$–computable, if and only if the canonically associated function $f' : Z \to C(X, Y)$ with $f'(z)(x) := f(z, x)$ is $(\delta'', [\delta \to \delta'])$–computable. As a direct consequence we obtain that matrices $A \in \mathbb{R}^{m\times n}$ can effectively be identified with linear mappings $f \in \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^m)$.

Lemma 2. *The correspondence between matrices and vector space homomorphisms is effective. This means that the mappings*

$$\mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}^{m\times n},\ \big(x \mapsto A \cdot x\big) \mapsto A$$
$$\mathbb{R}^{m\times n} \to \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^m),\ A \mapsto \big(x \mapsto A \cdot x\big)$$

are $\big([\rho^n \to \rho^m], \rho^{m\times n}\big)$*– and* $\big(\rho^{m\times n}, [\rho^n \to \rho^m]\big)$*–computable, respectively.*

Since a mapping like rank : $\mathbb{R}^{m\times n} \to \mathbb{R}$, which associates to each matrix $A \in \mathbb{R}^{m\times n}$ the dimension of its image, can only take finitely many different values, it is necessarily discontinuous and thus not $(\rho^{n\times m}, \rho)$–computable. Later on we will see that this mapping has a weaker computability property: given the matrix A we can at least find an increasing sequence of values which converge to $\mathrm{rank}(A)$. To express facts like this precisely, we will use two further representations $\rho_<, \rho_> :\subseteq \Sigma^\omega \to \mathbb{R}$. Roughly speaking, $\rho_<(p) = x$ if p is an (appropriately encoded) list of all rational numbers $q < x$. (Analogously, $\rho_>$ is defined with $q > x$.) It is a known fact that a mapping $f :\subseteq X \to \mathbb{R}$ is (δ, ρ)–computable, if and only if it is $(\delta, \rho_<)$– and $(\delta, \rho_>)$–computable [10].

Occasionally, we will also use some standard representation $\nu_\mathbb{N}, \nu_\mathbb{Q}$ of the natural numbers $\mathbb{N} = \{0, 1, 2, ...\}$ and the rational numbers $\mathbb{Q}$, respectively. Moreover, we will also need a representation for the space $\mathcal{L}^n$ of linear subspaces $V \subseteq \mathbb{R}^n$. Since all linear subspaces are non-empty closed spaces, we can use well-known representations of the hyperspace $\mathcal{A}^n$ of all closed non-empty subsets $A \subseteq \mathbb{R}^n$ (cf. [2, 10]). One way to represent such spaces is via the distance function $d_A : \mathbb{R}^n \to \mathbb{R}$, defined by $d_A(x) := \inf_{a\in A} d(x, a)$, where $d : \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ denotes the Euclidean metric of $\mathbb{R}^n$. Altogether, we define three representations $\psi^n, \psi^n_<, \psi^n_> :\subseteq \Sigma^\omega \to \mathcal{A}^n$. We let $\psi^n(p) = A$, if and only if $[\rho^n \to \rho](p) = d_A$. In other words, p encodes a set A w.r.t. ψ^n, if it encodes the distance function d_A w.r.t. $[\rho^n \to \rho]$. Analogously, let $\psi^n_<(p) = A$, if and only if $[\rho^n \to \rho_>](p) = d_A$ and let $\psi^n_>(p) = A$, if and only if $[\rho^n \to \rho_<](p) = d_A$. One can prove that $\psi^n_<$ encodes "positive" information about the set A (all open rational balls $B(q,r)$ which intersect A can be enumerated), and $\psi^n_>$ encodes "negative" information about A (all closed rational balls $\overline{B}(q,r)$ which do not intersect A can be enumerated). It is known that a mapping $f :\subseteq X \to \mathcal{A}^n$ is (δ,ψ^n)–computable, if and only if it is $(\delta,\psi^n_<)$– and $(\delta,\psi^n_>)$–computable [10]. We mention that

a) the operation $(f,A) \mapsto f^{-1}(A) \subseteq \mathbb{R}^n$ is $\big([\rho^n \to \rho^m],\psi^m_>,\psi^n_>\big)$–computable,
b) the operation $(f,B) \mapsto \overline{f(B)} \subseteq \mathbb{R}^m$ is $\big([\rho^n \to \rho^m],\psi^n_<,\psi^m_<\big)$–computable.

Together with Lemma 2 we obtain the following computability facts about kernel and image of matrices.

Lemma 3. *Given an $m \times n$ matrix A, its kernel can effectively be approximated from outside and its image can effectively be approximated from inside. More precisely, the mappings*

$$\ker : \mathbb{R}^{m\times n} \to \mathcal{A}^n,\ \ A \mapsto \big\{x \in \mathbb{R}^n : A\cdot x = 0\big\}$$
$$\operatorname{img} : \mathbb{R}^{m\times n} \to \mathcal{A}^m,\ A \mapsto \big\{A\cdot x : x \in \mathbb{R}^n\big\}$$

are $\big(\rho^{m\times n},\psi^n_>\big)$– and $\big(\rho^{m\times n},\psi^m_<\big)$–computable, respectively.

Finally, we will also use the notion of an *r.e. open* set $U \subseteq \mathbb{R}^{m\times n}$, which is a set such that there exists a $(\rho^{m\times n},\rho)$–computable function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ with $\mathbb{R}^{m\times n} \setminus U = f^{-1}\{0\}$. Given a representation δ of X, we will say more generally that a subset $U \subseteq Y \subseteq X$ is *δ–r.e. open* in Y, if $\delta^{-1}(U)$ is r.e. open in $\delta^{-1}(Y)$. Here a set $A \subseteq B \subseteq \Sigma^\omega$ is called *r.e. open* in B, if there exists some computable function $f :\subseteq \Sigma^\omega \to \Sigma^*$ with $\mathrm{dom}(f) \cap B = A$. Intuitively, a set U is δ–r.e. open in Y, if and only if there exists a Turing machine which halts for an input $x \in Y$ given w.r.t. δ, if and only if $x \in U$. It is known that a set $U \subseteq \mathbb{R}^{m\times n}$ is $\rho^{m\times n}$–r.e. open in $\mathbb{R}^{m\times n}$, if and only if it is r.e. open. If a set $U \subseteq X$ is δ–r.e. open in X, then we will say for short that it is *δ–r.e. open.*

3 Computing the Dimension

In this section we will discuss the problem to determine the dimension of a given linear subspace. We will see that in our setting this question is somehow related to the problem to determine a basis of a given linear space. First we note that we can compute the determinant of a matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ using the well-known formula $\det(A) = \sum_{\sigma\in\mathcal{S}_n} \mathrm{sign}(\sigma) \prod_{i=1}^n a_{\sigma(i)i}$, where $\mathcal{S}_n$ denotes the set of all permutations $\sigma : \{1,...,n\} \to \{1,...,n\}$. Since addition and multiplication are computable on the real numbers and by applying certain obvious closure schemes we can deduce:

Lemma 4. *Given an $n \times n$ matrix we can compute its determinant:*

$$\det : \mathbb{R}^{n\times n} \to \mathbb{R},\ A \mapsto \det(A)$$

is $\left(\rho^{n\times n}, \rho\right)$*–computable.*

Now we can test a tuple $(x_1, ..., x_n) \in \mathbb{R}^{m\times n}$ with $n \leq m$ for linear independence by searching for a non-zero determinant of the $n \times n$ sub-matrices of $(x_1, ..., x_n)$. Using this idea we can define a function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ by $f(A) := \sum_{S<A} |\det(S)|$, where the sum is over all $n \times n$ sub-matrices S of A. Thus f has the property that a tuple $(x_1, ..., x_n)$ is linearly independent, if and only if $f(x_1, ..., x_n) > 0$. Since f is computable by the previous lemma, this proves the following:

Lemma 5. *The property "linear independence", i.e. the set*

$$\{(x_1, ..., x_n) \in \mathbb{R}^{m\times n} : (x_1, ..., x_n) \text{ linearly independent } \},$$

is r.e. open.

Now we are prepared to investigate computability properties of the rank mapping $\text{rank} : \mathbb{R}^{m\times n} \to \mathbb{R}$ which maps a matrix A to the dimension of its image. Since the image of the rank mapping contains finitely many different values, the rank mapping cannot be continuous for $n, m \geq 1$. Nevertheless, we can approximate the rank from below. Using the previous lemma we can systematically search for a maximal linearly independent tuple among the column vectors of the matrix A and we can determine an increasing sequence of numbers $k \in \mathbb{N}$ in this way which converges to $\text{rank}(A)$.

Proposition 6. *The rank of a matrix can be approximated from below:*

$$\text{rank} : \mathbb{R}^{m\times n} \to \mathbb{R},\ A \mapsto \text{rank}(A)$$

is $\left(\rho^{m\times n}, \rho_<\right)$*–computable, but neither* $\left(\rho^{m\times n}, \rho_>\right)$*–computable, nor continuous.*

The linear span mapping which maps a matrix A to the linear span of its column vectors has similar properties as the rank mapping. If we know a basis of a linear subspace, we can even obtain complete information about its span.

Proposition 7. *The linear span mapping*

$$\text{span} : \mathbb{R}^{m\times n} \to \mathcal{A}^m, (x_1, ..., x_n) \mapsto \bigcap \left\{V \in \mathcal{L}^m :\ x_1, ..., x_n \in V\right\}$$

is $(\rho^{m\times n}, \psi^m_<)$*–computable, but neither* $(\rho^{m\times n}, \psi^m_>)$*–computable nor continuous. Restricted to linear independent inputs* $(x_1, ..., x_n) \in \mathbb{R}^{m\times n}$ *the linear span mapping is even* $(\rho^{m\times n}, \psi^m)$*–computable.*

Proof. The first property follows directly from Lemma 3 since $\mathrm{span}(A) = \mathrm{img}(A)$. It is easy to see that the linear span mapping is not continuous in the zero matrix $A = 0$. Finally, we have to prove that the linear span mapping is $(\rho^{m\times n}, \psi_>^m)$–computable, restricted to linear independent inputs. If $n = m$, then this is trivial since $\mathrm{span}(x_1, ..., x_m) = \mathbb{R}^m$ for linearly independent $(x_1, ..., x_m)$. For the case $n < m$ we will again use the function $f : \mathbb{R}^{m\times(n+1)} \to \mathbb{R}$, defined by $f(A) := \sum_{S<A} |\det(S)|$, where the sum is over all $(n+1)\times(n+1)$ sub-matrices S of A. As we have seen, f is computable and $(x_1, ..., x_n, x)$ is linearly independent, if and only if $f(x_1, ..., x_n, x) \neq 0$. Especially, we can deduce

$$\mathrm{span}(x_1, ..., x_n) = \{x \in \mathbb{R}^m : f(x_1, ..., x_n, x) = 0\},$$

provided that $(x_1, ..., x_n)$ is linearly independent. Using type conversion we can show that $g : \mathbb{R}^{m\times n} \to C(\mathbb{R}^m, \mathbb{R})$ with $g(x_1, ..., x_n)(x) := f(x_1, ..., x_n, x)$ is $(\rho^{m\times n}, [\rho^m \to \rho])$–computable. Moreover, as we have already stated, it is known that $C(\mathbb{R}^m, \mathbb{R}) \to \mathcal{A}^m, h \mapsto h^{-1}\{0\}$ is $([\rho^m \to \rho], \psi_>^m)$–computable. Thus, span, restricted to linearly independent inputs, is $(\rho^{m\times n}, \psi_>^m)$–computable. $\square$

As we have seen, the dimension of a linear subspace cannot be computed from a tuple of vectors whose linear span generates the subspace. In general the information included in such a tuple does not suffice to determine upper bounds on the dimension. If we additionally supply negative information about the linear subspace, the dimension operator becomes computable. As a preparation we first prove that the set with the zero-space as single point is r.e. open in the set of linear subspaces w.r.t. negative information.

Lemma 8. *The set $\{\{0\}\} \subseteq \mathcal{A}^n$ is $\psi_>^n$–r.e. open in $\mathcal{L}^n$.*

Proof. We note that it is known that given a $\psi_>^n$–name p of a subspace V, we can effectively find a $[\nu_{\mathbb{N}} \to \nu_{\mathbb{Q}}^2]$–name q of a function $f : \mathbb{N} \to \mathbb{Q}^2$ such that $\mathbb{R}^n \setminus V = \bigcup_{k=0}^{\infty} B(c_k, r_k)$, where $(c_k, r_k) := f(k)$ (cf. [2]). In other words, we can effectively represent V by a sequence of open rational balls, whose union exhausts the exterior of V. For a set $V \in \mathcal{L}^n$ we obtain

$$V = \{0\} \iff V \cap S^{n-1} = \emptyset,$$

where $S^{n-1} := \partial B(0, 1) = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_1^2 + ... + x_n^2 = 1\}$ denotes the unit sphere of the n-dimensional space $\mathbb{R}^n$. Since S^{n-1} is compact, we can conclude

$$V \cap S^{n-1} = \emptyset \iff (\exists m)\ S^{n-1} \subseteq \bigcup_{k=0}^{m} B(c_k, r_k).$$

Since S^{n-1} is a recursive compact set, i.e. there exists some computable s such that $\psi^n(s) = S^{n-1}$, we can deduce that we can even effectively find an m with the property above, if such an m exists. This proves that there exists a computable function $f :\subseteq \Sigma^\omega \to \Sigma^*$ such that $p \in \mathrm{dom}(f)$, if and only if $\psi_>^n(p) = \{0\}$, for all $p \in (\psi_>^n)^{-1}(\mathcal{L}^n)$. $\square$

No we are prepared to prove the following main result of our paper.

Theorem 9. *One can compute the dimension of a given linear subspace:*

$$\dim :\subseteq \mathcal{A}^n \to \mathbb{R},\ V \mapsto \dim(V)$$

is $\left(\psi^n_<, \rho_<\right)$*– and* $\left(\psi^n_>, \rho_>\right)$*–computable. It is in particular* $\left(\psi^n, \rho\right)$*–computable.*

Proof. First of all, it is known that given a $\psi^n_<$–name p of a subspace V, we can effectively find a $[\nu_{\mathbb{N}} \to \rho^n]$–name q of a function $f : \mathbb{N} \to \mathbb{R}^n$ such that the image $f(\mathbb{N})$ is dense in V (cf. [2]). We claim that if $\dim(V) = k$, then there exists a tuple $(i_1, ..., i_k) \in \mathbb{N}^k$ such that $(f(i_1), ..., f(i_k))$ is a basis of V. If $(x_1, ..., x_k) \in \mathbb{R}^{n\times k}$ is an arbitrary basis of V, then there exist an open neighbourhood U of $(x_1, ..., x_k)$ which only consists of tuples of linear independent vectors (by Lemma 5 linear independence especially is an open property). Since $f(\mathbb{N})$ is dense in V, there exists a tuple $(i_1, ..., i_k) \in \mathbb{N}^k$ such that $(f(i_1), ..., f(i_k)) \in U \cap V^k$. This proves the claim. Thus, we can use the name q to search for tuples $(i_1, ..., i_m) \in \mathbb{N}^m$ of maximal size m such that $(f(i_1), ..., f(i_m))$ is linear independent. In this way we can produce an increasing sequence $m_1, m_2, ...$ of natural numbers. Lemma 5 guarantees that the whole procedure is effective and the claim proved before guarantees that the sequence converges to $\dim(V)$.

We note that it is known that the intersection operation $\mathcal{A}^n \times \mathcal{A}^n \to \mathcal{A}^n$, $(A, B) \mapsto A \cap B$ is $([\psi^n_>, \psi^n_>], \psi^n_>)$–computable (cf. [10]). If we can find some linear subspace U of $\mathbb{R}^n$ with $U \cap V = \{0\}$ and $\dim(U) = n - m$, then we can conclude $\dim(V) + \dim(U) = \dim(U \oplus V) \leq n$ and thus $\dim(V) \leq m$. On the other hand, if $\dim(V) \leq m$, then there exists always such a subspace U. Thus, to guarantee $\dim(V) \leq m$, it suffices to find $n - m$ linear independent vectors $x_1, ..., x_{n-m}$ such that $\mathrm{span}(x_1, ..., x_{n-m}) \cap V = \{0\}$. If such vectors exist, then there exist also rational vectors $x_1, ..., x_{n-m} \in \mathbb{Q}^n$ with the same property, since linear independence is an open property by Lemma 5, the test on equality with $\{0\}$ is open by Lemma 8, the linear span mapping (by Proposition 7) and intersection are continuous (w.r.t. the final topology of $\psi^n_>$). Thus, we can produce a decreasing sequence of natural number $m_1, m_2, ...$ by searching for a minimal m and linear independent rational vectors $x_1, ..., x_{n-m} \in \mathbb{Q}^n$ with the property above. The considerations above together with Lemma 5, Proposition 7 and Lemma 8 show that the whole procedure is effective and that the sequence converges to $\dim(V)$. $\square$

Using the same method as in the proof before, we can construct a basis of a linear subspace as a set w.r.t. $\psi^n_<$. If we know the dimension in advance, then by virtue of Lemma 5 we can determine a basis directly by representing its vectors.

Corollary 10. *One can effectively find a basis of a given linear subspace. More precisely, the multi-valued mapping*

$$\mathrm{basis} :\subseteq \mathcal{A}^m \rightrightarrows \mathcal{A}^m,\ V \mapsto \left\{\{b_1, ..., b_n\} \subseteq \mathbb{R}^m : (b_1, ..., b_n) \textit{ basis of } V\right\}$$

is $\left(\psi^m_<, \psi^m_<\right)$*– and* $\left(\psi^m, \psi^m\right)$*–computable. If* $n := \dim(V)$ *is known in advance,*

$$\mathrm{basis}' :\subseteq \mathcal{A}^m \rightrightarrows \mathbb{R}^{m\times n},\ V \mapsto \{(b_1, ..., b_n) \in \mathbb{R}^{m\times n} : (b_1, ..., b_n) \textit{ basis of } V\}$$

is even $\left(\psi^m_<, \rho^{m\times n}\right)$*–computable.*

We can also effectively compute complementary spaces.

Corollary 11. *Given a linear subspace $V \subseteq \mathbb{R}^m$, one can effectively find a linear subspace $U \subseteq \mathbb{R}^m$ such that $V \oplus U = \mathbb{R}^m$. More precisely, the multi-valued mapping*

$$\mathrm{compl} :\subseteq \mathcal{A}^m \rightrightarrows \mathcal{A}^m,\ V \mapsto \{U \in \mathcal{L}^m : V \oplus U = \mathbb{R}^m\}$$

is (ψ^m, ψ^m)–computable.

4 Conclusion

Computations with sets find an increasing interest in computable analysis, cf. for instance [3, 7, 12, 11, 2]. One of the motivations is to find suitable data structures for solid modeling and other practical applications of computations with sets [7]. We have presented a result which shows that the dimension of a linear subspace can be computed, provided that the subspace is represented via its distance function. This result can be considered as a starting point of "computable linear algebra". Many interesting questions in this field still remain open.

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