THE EMPEROR'S NEW RECURSIVENESS: THE EPIGRAPH OF THE EXPONENTIAL FUNCTION IN TWO MODELS OF COMPUTABILITY

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In his book "The Emperor's New Mind" Roger Penrose implicitly defines some criteria which should be met by a reasonable notion of recursiveness for subsets of Euclidean space. We discuss two such notions with regard to Penrose's criteria: one originated from computable analysis, and the one introduced by Blum, Shub and Smale.

1 Introduction

In his book "The Emperor's New Mind" Roger Penrose [1] raises the question whether the famous Mandelbrot set $M \subseteq \mathbb{R}^2$ can be considered as recursive in some well-defined sense. Throughout his discussion of this problem Penrose uses an intuitive notion of recursiveness and he complains about the lack of a mathematically precise meaning of this notion. On the one hand, he argues that it is insufficient to define recursiveness of a set as decidability with respect to computable points, since in this case even a simple set like the unit ball $B := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ does not become recursive. Since Penrose is convinced that the unit ball should become recursive we are led to introduce the following criterion.

Penrose's first criterion. A reasonable notion of recursiveness for subsets of Euclidean space should make the closed unit ball recursive.

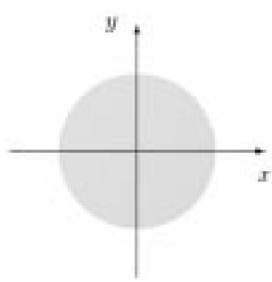

Figure 1. The closed unit ball B

On the other hand, Penrose argues that certain other ways to define recursiveness are also inappropriate, especially, because they do not handle the border of the sets under consideration in the right way. This aspect is important since the complexity of sets is often inherent in their border, as in case of Mandelbrot's set. For instance, a definition of recursiveness as decidability with respect to rational or algebraic numbers is insufficient, since in this case sets like the closed epigraph of the exponential function $E := \{(x, y) \in \mathbb{R}^2 : y \geq e^x\}$ would not be handled appropriately. The border of this set does not contain any algebraic point besides $(0, 1)$ and thus the border is irrelevant to a decision procedure which is restricted to algebraic points. Of course, Penrose is convinced that a set, easily structured like the closed epigraph of the exponential function, should be recursive. This motivates the second criterion.

Penrose's second criterion. A reasonable notion of recursiveness for subsets of Euclidean space should make the closed epigraph of the exponential function recursive.

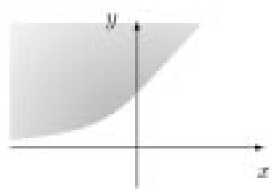

Figure 2. The closed epigraph E of the exponential function

Apparently, there are several similar conditions and Penrose's criteria are by no means sufficient conditions for a reasonable notion of recursiveness. They are just necessary conditions; a notion of recursiveness which does not meet Penrose's criteria would be highly suspicious since it could be doubted whether it reflects algorithmic complexity in the right way. Since Penrose did not present any notion which fulfills all his requirements, it seems as if there exists no suitable notion of recursiveness.

The aim of this paper is to compare two existing notions of recursiveness for subsets of Euclidean space and to find out which comes closest to Penrose's requirements. The first notion is based on computable analysis and has been developed and investigated by several authors. The basic idea of recursive analysis is to call a function $f : \mathbb{R}^n \to \mathbb{R}$ *computable*, if there exists a Turing machine which transforms Cauchy sequences of rationals, rapidly converging to an input x, into Cauchy sequences of rationals, rapidly converging to the output $f(x)$. Moreover, a set $A \subseteq \mathbb{R}^n$ is called *recursive*, if its *distance function* $d_A : \mathbb{R}^n \to \mathbb{R}$ is computable.[a] Here d denotes the Euclidean metric. This notion of recursiveness straightforwardly generalizes the notion of recursiveness from classical computability theory (see Odifreddi [3]): if we endow the natural numbers $\mathbb{N}$ with the discrete metric, then the distance function of a subset $A \subseteq \mathbb{N}$ is equal to its characteristic function and computability of the characteristic function is equivalent to recursiveness of the set A. In Euclidean space the distance function is a "continuous substitute" for the characteristic function. Although recursiveness of subsets of Euclidean space in this sense does not correspond to the intuition of "decidability", it is a formal generalization of the classical notion of recursiveness. Especially, a subset $A \subseteq \mathbb{N}$ considered as a subset of $\mathbb{R}$ is recursive, if and only if it is classically decidable. Finally, it is easy to prove that this notion of recursiveness meets Penrose's criteria.

As a second notion of recursiveness we will investigate the notion which has been developed by Blum, Shub and Smale.[4,5] In their theory a function $f : \mathbb{R}^n \to \mathbb{R}$ is computable (we will call it *algebraically computable* for the following), if there exists a real random access machine which computes f. Such a machine uses real number registers, arbitrary constants, arithmetic operations, comparisons and equality tests. Moreover, a set $A \subseteq \mathbb{R}^n$ is called recursive by Blum, Shub and Smale (we will call it *algebraically recursive* for the following), if its characteristic function is algebraically computable. If we restrict the class of constants appropriately (for instance to rational numbers), then a set $A \subseteq \mathbb{N}$ considered as a subset of $\mathbb{R}$ is algebraically recursive, if and only if it is classically decidable. In this sense the notion of algebraic recursiveness is a second generalization of the classical notion of recursiveness. Obviously, the unit ball is algebraically recursive and hence Penrose's first criterion is met. Blum and Smale [6] have proved that Mandelbrot's set is not algebraically recursive and hence it seems as if they have given an answer to Penrose's original question. But with a similar technique we will prove that the closed epigraph of the exponential function is not algebraically recursive and hence it is highly questionable whether Blum and Smale's answer to Penrose's question is significant. If even a simple set like the epigraph of the exponential function is not algebraically recursive, we can conclude that algebraic non-recursiveness obviously does not reflect the intrinsic algorithmic complexity of a set.

[a] The idea of using distance functions to characterize "located" sets has first been used in constructive analysis, see Bishop and Bridges.[2]

2 Recursive and Recursively Enumerable Sets

In this section we give the precise definitions of several classes of recursively enumerable and recursive sets and we will give a short survey on some elementary properties.

Let $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the *Euclidean metric* of $\mathbb{R}^n$, defined by $d(x, y) := \sqrt{\sum_{i=1}^n |x_i - y_i|^2}$ for all $x, y \in \mathbb{R}^n$. By $B(x, r) := \{y \in \mathbb{R}^n : d(x, y) < r\}$ we denote the *open balls* and by $\overline{B}(x, r) := \{y \in \mathbb{R}^n : d(x, y) \leq r\}$ the *closed balls* with respect to d. For each set $A \subseteq \mathbb{R}^n$ we denote by $d_A : \mathbb{R}^n \to \mathbb{R}$ the *distance function* of A, defined by $d_A(x) := \inf_{a \in A} d(x, a)$. Let $\alpha : \mathbb{N} \to \mathbb{R}^n$ be some standard enumeration with $\text{range}(\alpha) = \mathbb{Q}^n$, defined for instance by $\alpha\langle\langle i_1, j_1, k_1\rangle, ..., \langle i_n, j_n, k_n\rangle\rangle := (\frac{i_1 - j_1}{k_1 + 1}, ..., \frac{i_n - j_n}{k_n + 1})$. Here, $\langle . \rangle : \mathbb{N}^2 \to \mathbb{N}$ denotes *Cantor's Pairing Function*, defined by $\langle i, j\rangle := \frac{1}{2}(i + j)(i + j + 1) + j$, which can inductively be extended to a function $\langle . \rangle : \mathbb{N}^n \to \mathbb{N}$. All these pairing functions are bijective and computable, as well as their inverses.

We assume that the reader is familiar with the definition of computable real functions (see, for instance, Weihrauch,[7] Pour-El and Richards,[8] Ko [9]). We briefly recall the ideas: a function $f :\subseteq \mathbb{R}^n \to \mathbb{R}$ is called *computable*, if there exists a Turing machine which transforms each Cauchy sequence $(q_i)_{i\in\mathbb{N}}$ of rational numbers $q_i \in \mathbb{Q}^n$ (encoded with respect to α), which rapidly converges to some $x \in \text{dom}(f)$ into a Cauchy sequence $(r_i)_{i\in\mathbb{N}}$ of rational numbers $r_i \in \mathbb{Q}$, which rapidly converges to $f(x)$. Here, rapid convergence means $d(q_i, q_k) \leq 2^{-k}$ for all $i > k$ (and correspondingly for $(r_i)_{i\in\mathbb{N}}$). Of course, a Turing machine which transforms an infinite sequence into an infinite sequence has to compute infinitely long, but in the long run the correct output sequence has to be produced. It is reasonable to assume one-way output tapes for such machines since otherwise the output after some finite time would be useless (because it could be replaced later).

Functions, such as $\exp, \sin, \cos, \ln$ and $\max$ are examples of computable functions. One of the basic observations of computable analysis is that computable functions are continuous. This is because approximations of the output are computed from approximations of the input and therefore each approximation of the output has to depend on some approximation of the input. Computable functions of type $f : \mathbb{N} \to \mathbb{R}^n$ can be defined similarly and are called *computable sequences*.

Now we are prepared to define the notion of recursively enumerable and recursive subsets in the sense of computable analysis (see Brattka and Weihrauch [10] for a survey). These notions are explicitly defined for open or closed sets, respectively.

Definition 2.1 (Recursively enumerable open and closed sets)

1. An open subset $A \subseteq \mathbb{R}^n$ is called *recursively enumerable*, (r.e. for short), if there is a computable function $f : \mathbb{N} \to \mathbb{N}^2$ such that $A = \bigcup_{(i,j)\in\mathrm{range}(f)} B(\alpha(i), 2^{-j})$.

2. A closed subset $A \subseteq \mathbb{R}^n$ is called *recursively enumerable*, (r.e. for short), if $A = \emptyset$ or there is a computable sequence $f : \mathbb{N} \to \mathbb{R}^n$ such that $\mathrm{range}(f)$ is dense in A.

3. An open (closed) set is called *co-recursively enumerable* (co-r.e. for short), if its complement A^{c} is r.e.

4. An open (closed) set is called *recursive*, if it is r.e. and co-r.e.

Recursively enumerable open sets have first been introduced and investigated by Lacombe.[11] Equivalent definitions to the given ones have been investigated by several authors (see Weihrauch and Kreitz,[12,13] Ko et al.,[14,9,15] Ge and Nerode,[16] Zhou,[17] Zhong,[18] Brattka [19]). The following characterization gives an impression of the stability of the definition of r.e. sets. For completeness we also mention the characterizations via semi-computable distance functions. These notions are not used any further in this paper and the interested reader is refered to Brattka and Weihrauch [10] for the definitions and proofs.

Lemma 2.2 (Characterization of r.e. closed sets) *Let $A \subseteq \mathbb{R}^n$ be a closed set. Then the following equivalences hold:*

1.
$$\begin{aligned} & A \text{ is recursively enumerable} \\ \iff & \{(i,j) \in \mathbb{N}^2 : A \cap B(\alpha(i), 2^{-j}) \neq \emptyset\} \text{ is recursively enumerable} \\ \iff & d_A : \mathbb{R}^n \to \mathbb{R} \text{ is upper semi-computable,} \end{aligned}$$

2.
$$\begin{aligned} & A \text{ is co-recursively enumerable} \\ \iff & \{(i,j) \in \mathbb{N}^2 : A \cap \overline{B}(\alpha(i), 2^{-j}) = \emptyset\} \text{ is recursively enumerable} \\ \iff & d_A : \mathbb{R}^n \to \mathbb{R} \text{ is lower semi-computable} \\ \iff & A = f^{-1}\{0\} \text{ for some computable function } f : \mathbb{R}^n \to \mathbb{R}, \end{aligned}$$

3.
$$A \text{ is recursive} \iff d_A : \mathbb{R}^n \to \mathbb{R} \text{ is computable.}$$

Using these characterizations and the fact that the exponential function is a computable function one can easily show that the notion of recursiveness of computable analysis fulfills Penrose's criteria.

Proposition 2.3 (Recursive sets and Penrose's criteria)

1. *The closed unit ball* $B := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ *is a recursive set.*

2. *The closed epigraph* $E := \{(x,y) \in \mathbb{R}^2 : y \geq e^x\}$ *is a recursive set.*

Proof.

1. We obtain $d_B(x,y) = \max\{0, \sqrt{x^2+y^2} - 1\}$ for the distance function $d_B : \mathbb{R}^2 \to \mathbb{R}$. Thus, d_B is computable and B is recursive.

2. There exists a computable function $f : \mathbb{N} \to \mathbb{R}^2$ such that $\text{range}(f) = \{(x, e^x + y) \in \mathbb{R}^2 : x, y \in \mathbb{Q}, y \geq 0\}$, since the exponential function is computable. Since $\text{range}(f)$ is dense in E it follows that E is an r.e. closed set. The function $g : \mathbb{R}^2 \to \mathbb{R}$ with $g(x,y) := \max\{0, e^x - y\}$ is computable and $E = g^{-1}\{0\}$. Thus, E is a co-r.e. closed set. Altogether, E is a recursive closed set. □

More generally, the proof of 2. shows that the closed epigraph $\text{epi}(f) = \{(x,y) \in \mathbb{R}^2 : y \geq f(x)\}$ of a computable function $f : \mathbb{R} \to \mathbb{R}$ is a recursive set.[b] It is worth noticing that the notion of computability and the notion of recursiveness of computable analysis fit together very well: a continuous function $f : \mathbb{R} \to \mathbb{R}$ is computable, if and only if its graph is recursive as a closed subset of $\mathbb{R}^2$ (see Weihrauch [20]).

3 Algebraic Recursiveness

In this section we want to prove that the notion of algebraic recursiveness does not meet Penrose's second criterion. We start with the definition of algebraically r.e. sets as halting sets of real random access machines, as they have been used by Blum, Shub and Smale.[4,5] These real random access machines use real number registers, arbitrary constants, arithmetic operations, comparisons and equality tests. We assume that the reader is familiar with the precise definitions. From the point of view of computable analysis especially the comparisons and equality tests are problematic. From the point of view of classical computability theory also the constants are suspicious since one can code an arbitrary function $f : \mathbb{N} \to \mathbb{N}$ in such a constant.

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[b] For a general discussion of computability properties of the epigraph, see Zheng et al.[20,21]

Definition 3.1 (Algebraically r.e. sets) Let $A \subseteq \mathbb{R}^n$.

1. A is called *algebraically r.e.*, if A is the halting set of some real random access machine.

2. A is called *algebraically recursive*, if A as well as its complement A^c is algebraically r.e.

If A is the halting set of a real random access machine which does only use rational constants, then we will say that A is *algebraically r.e. with rational constants*. Obviously, the unit ball $B := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ is an algebraically recursive set, even with rational constants. We just have to compute $x^2 + y^2$ and test $x^2 + y^2 \leq 1$.

Proposition 3.2 *The closed unit ball* $B := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ *is algebraically recursive with rational constants.*

One can easily prove that the *open* epigraph of the exponential function $\{(x, y) \in \mathbb{R}^2 : y > e^x\}$ is an r.e. open set and hence it is also algebraically r.e. (as any other r.e. open set). On the other hand, we will show that the closed epigraph E of the exponential function is not algebraically recursive. Indeed, we will prove that it is not even algebraically r.e. The proof uses some standard techniques of Blum, Shub and Smale's theory, especially their Path Decomposition Theorem, which states that each algebraically r.e. set is a countable (disjoint) union of basic semi-algebraic sets (see Blum et al.[5]).

We recall some basic definitions and facts from real algebraic geometry (which can be found in Bochnak et al.[22] and Marker et al.[23]). The class of *semi-algebraic* subsets of $\mathbb{R}^n$ is the smallest class of subsets of $\mathbb{R}^n$ which contains all sets $\{x \in \mathbb{R}^n : p(x) > 0\}$ with real polynomials $p : \mathbb{R}^n \to \mathbb{R}$, and which is closed under finite intersection, finite union and complement. Each semi-algebraic set can be written as finite union of *basic semi-algebraic sets*, which have the form

$$\{x \in \mathbb{R}^n : p_1(x) = 0, ..., p_i(x) = 0, q_1(x) > 0, ..., q_j(x) > 0\},$$

where $p_1, ..., p_i, q_1, ..., q_j : \mathbb{R}^n \to \mathbb{R}$ are real polynomials. A (partial) function $f :\subseteq \mathbb{R}^n \to \mathbb{R}$ is called *semi-algebraic*, if its *graph*

$$\mathrm{graph}(f) := \{(x, y) \in \mathbb{R}^{n+1} : f(x) = y\}$$

is a semi-algebraic set. Using the normal form given above, it is easy to show that each semi-algebraic function is *algebraic*, i.e. there exists some real polynomial $p : \mathbb{R}^{n+1} \to \mathbb{R}$, $p \neq 0$ such that $p(x, f(x)) = 0$ for all $x \in \mathrm{dom}(f)$. By the Theorem of Tarski-Seidenberg semi-algebraic sets are closed under projection and one can conclude that the *interior* A°, the *closure* $\overline{A}$ and hence the *border* $\partial A = \overline{A} \setminus A^\circ$ of a semi-algebraic set A is semi-algebraic too (additionally, one uses the fact that the Euclidean metric is a semi-algebraic function). Correspondingly, one can see that the *lower border*

$$A^{\downarrow} := \{(x, y) \in \mathbb{R}^2 : (x, y) \in A \text{ and } (\forall z \in \mathbb{R})((x, z) \in A \Longrightarrow z \geq y)\}$$

is semi-algebraic if $A \subseteq \mathbb{R}^2$ is. By $f|_U$ we will denote the *restriction* of a function f with $\mathrm{dom}(f|_U) = \mathrm{dom}(f) \cap U$. Now we are prepared to prove the following result.

Proposition 3.3 *The closed epigraph* $E := \{(x, y) \in \mathbb{R}^2 : y \geq e^x\}$ *of the exponential function is not algebraically r.e.*

Proof. Let $E := \{(x, y) \in \mathbb{R}^2 : y \geq f(x)\}$ be the closed epigraph of the exponential function $f : \mathbb{R} \to \mathbb{R}$. Let us assume that E is algebraically r.e. Then, by the Path Decomposition Theorem, E is a countable union of semi-algebraic sets $A_i \subseteq \mathbb{R}^2$, i.e. $E = \bigcup_{i=0}^{\infty} A_i$. Since the closure of a semi-algebraic set is semi-algebraic too, we can assume w.l.o.g. that all sets A_i are closed. Especially, we obtain $\partial E = \bigcup_{i=0}^{\infty} (\partial E \cap A_i)$ and since the border ∂E is a complete subspace of $\mathbb{R}^2$ it follows by Baire's Category Theorem that there is some $i \in \mathbb{N}$ and a non-empty open set $U \subseteq \mathbb{R}^2$ such that $\emptyset \neq \partial E \cap U \subseteq A_i$. Since $\partial E = \mathrm{graph}(f)$ and f is continuous, there are some non-empty open intervals $I, J \subseteq \mathbb{R}$ such that $f(I) \subseteq J$ and $V := I \times J \subseteq U$. Hence $\mathrm{graph}(f|_I) = \partial E \cap V = A_i^{\downarrow} \cap V$ is semi-algebraic, since $A_i^{\downarrow}$ and V are semi-algebraic. But using the Identity Theorem for real-analytic functions and the power series expansion of the exponential function, one can prove that $f|_I$ is not algebraic. Contradiction! $\square$

This proposition proves that algebraic recursiveness does not meet Penrose's second criterion. We will call a function $f : \mathbb{R} \to \mathbb{R}$ *everywhere transcendental*, if $f|_U$ is not algebraic for each non-empty open set $U \subseteq \mathbb{R}$. The proof that the exponential function is everywhere transcendental can be found in basic texts on analysis (see, for instance, Erwe [24]). Besides the fact that the exponential function is everywhere transcendental and continuous, we have not used any specific properties of the exponential function in the previous proof. By symmetry we obtain the following general result.

Theorem 3.4 *If* $f : \mathbb{R} \to \mathbb{R}$ *is an everywhere transcendental and continuous function, then neither the closed epigraph, nor the closed hypograph, nor the graph of* f *is algebraically r.e.*

It is worth noticing that the notions of algebraic recursiveness and algebraic computability do not fit together in the same sense as the notions of recursiveness and computability of computable analysis. The square root function $f :\subseteq \mathbb{R} \to \mathbb{R}, x \mapsto \sqrt{x}$ is an example of a function which is not algebraically computable but whose graph is algebraically recursive. Hence, the algebraic non-recursiveness of the graph of the exponential function cannot simply be deduced from the fact that the exponential function is not algebraically computable.[c]

4 Conclusion

We have seen that the notion of algebraic recursiveness does not meet Penrose's criteria, while the notion of recursiveness from computable analysis does. The latter notion describes recursiveness in terms of computability of the distance function d_A of a set A. In view of the fact that equality on the real numbers is undecidable, recursiveness in this sense is the best what one could expect. Recursiveness implies "decidability up to the equality test on the real numbers": if we only could decide whether $d_A(x) = 0$, then we could decide whether $x \in A$ or not.

An essential question remains open. We do not know whether the Mandelbrot set is a recursive closed set or not. It is easy to see that it is a co-r.e. closed set but it is still a challenging open question to find out whether it is also an r.e. closed set or not!

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[c] Over algebraically closed fields a function is algebraically computable, if and only if its graph is algebraically recursive, see Ceola and Lecomte.[25]

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