Turing Computability of (Non-)Linear Optimization*

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Abstract

We consider the classical LINEAR OPTIMIZATION problem, but in the Turing rather than the REAL-RAM model. Asking for mere computability of a function's maximum over some closed domain, we show that the common presumptions 'full-dimensional' and 'bounded' in fact cannot be omitted: The sound framework of Recursive Analysis enables us to rigorously prove this folkloristic observation! On the other hand, convexity of this domain may be weakened to connectedness, and even *non*-linear functions turn out to be effectively optimizable.

1 Motivation

The gap between (theoretical) algorithm design and (practical) implementation reveals a particular challenge in Computational Geometry: Provably correct algorithms keep, upon implementation, reporting not only inaccurate but sometimes even entirely invalid results, especially upon input of degenerate configurations [9] — for obvious reasons: Algorithms in Computational Geometry are most generally developed in the REAL-RAM model [1], capable of operating on real numbers *exactly*. Actual digital computers however can process in each step only a finite amount of information [2].

We therefore find it necessary to consider a different model of real number computation. In fact, Alan Turing himself introduced 'his' machine in order to study computability aspects over $\mathbb{R}$ [14] and thereby initiated the nowadays well-established field of RECURSIVE ANALYSIS [8, 12, 15]. Roughly speaking, a Turing machine is said to compute the real number r if it can output rational approximations of arbitrarily prescribable precision.

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It is straight forward to adopt this model to Computational Geometry. In fact [6] already did so in investigating Turing computability of the extreme points of a convex body, a continuous generalization of classical convex hull problems. [11] considers geometric predicates on convex bodies such as MEMBERSHIP and LINEINTERSECT in exactly the same sense. Also Grötschel, Lovász, and Schrijver in their enlightening book [7] developed polynomial-time Turing-algorithms for CONVEX OPTIMIZATION up to prescribable error $\varepsilon > 0$: although not mentioned explicitly, their WEAK OPTIMIZATION PROBLEM is exactly in the spirit of Recursive Analysis.

The present work generalizes a fundamental result from [7]: Dropping the requirement of *polynomial* time complexity, we present Turing algorithms for the optimization of arbitrary continuous objective functions. Even the convexity condition may be relaxed: It is shown that one can computably optimize over compact, connected, and everywhere full-dimensional domains, so-called 'bodies'. To this end, we establish new computability results for operators on bodies such as union, intersection, and function pre-image. The latter considerations resemble similar ones in [5] which, however, were performed in the domain-theoretic setting.

2 Introduction

Recursive Analysis combines Classical Analysis with discrete Recursion Theory: Results of the former such as INTERMEDIATE VALUE or FIXED POINT Theorems claiming *existence* of certain objects are investigated with respect to the *computability* of these objects [13, 10, 16, 18]. A real number $r \in \mathbb{R}$ for example is computable if there exists some Turing machine which outputs two sequences p_n and q_n of integers such that $|r - p_n/q_n| < 2^{-n}$.

Obviously, every rational number is computable in this sense; and so are some irrationals such as π, e, $\sqrt{2}$. On the other hand not *every* of the uncountable many real numbers can be computed by only countable many Turing machines: any non-recursive subset N of $\mathbb{N}$ gives rise to a non-computable real $\sum_{n\in N} 2^{-n}$.

As usual, Turing machines become able to compute non-numeric objects (e.g., graphs) by *encoding* them into binary strings and operating on these 'names': Following the general framework of TYPE-2 THEORY OF EFFECTIVITY [15], let X be an arbitrary set not exceeding continuum cardinality $|X| \leq \mathfrak{c}$ and $\alpha :\subseteq \Sigma^{\mathbb{N}} \to X$ a fixed partial surjective mapping, a so-called representation. Machine M is said to α-compute $x \in X$ if it outputs an α-name for x, that is, an infinite string $\bar{\sigma} \in \Sigma^{\mathbb{N}}$ such that $\alpha(\bar{\sigma}) = x$. For infinite outputs to make sense after finite time, the output tape is required to be *one-way*.

In the above example $X = \mathbb{R}$, let $\rho(\bar{\sigma})$ be defined for all strings of the form $\bar{\sigma} = \big(\text{bin}(p_1), \text{bin}(q_1), \text{bin}(p_2), \text{bin}(q_2), \ldots\big)$ where $r := \lim_{n\to\infty}(p_n/q_n)$ exists and $|r - p_n/q_n| < 2^{-n}$; here $\text{bin}(i)$ denotes usual binary encoding of integer i. This $\rho : \bar{\sigma} \mapsto r$ is the standard representation of real numbers.

Different representations of the same set X may induce different notions of computability. For instance define $\rho_<(\bar{\sigma})$ to be the real number r iff $\bar{\sigma}$ is (an encoding of) a sequence of rationals converging to r *from below*. It is easy to see that any ρ-computable real is $\rho_<$-computable as well, but not vice versa. If however some $r \in \mathbb{R}$ is both $\rho_<$-computable and $\rho_>$-computable — the latter defined analogously by convergence from above — then r must be ρ-computable, too.

The preceding considerations dealt with computability of single objects x. This was reflected by the fact that the corresponding machine M received no input but had to output a name σ for x 'from scratch'. Let us now fix two sets X and Y with corresponding representations α and β. A function $f : X \to Y$ is (α, β)-computable if some Turing machine, upon input of any α-name $\bar{\sigma}$ for $x \in X$, outputs a β-name $\bar{\tau}$ for $y = f(x)$.

The MAIN THEOREM OF RECURSIVE ANALYSIS now states that any (ρ, ρ)-computable real function must necessarily be continuous! More generally: Let $\Sigma^{\mathbb{N}}$ be equipped with the Cantor Topology and $\alpha :\subseteq \Sigma^{\mathbb{N}} \to X$, $\beta :\subseteq \Sigma^{\mathbb{N}} \to Y$ admissible representations, (i.e., 'well-behaved' w.r.t. the final topologies on X and Y). If $f : X \to Y$ is (α, β)-computable, then f is continuous: see Theorem 3.2.11 in [15].

The issue of non-/uniform computability is best illustrated by comparing the following two statements:

Heredity: x α-computable $\Rightarrow$ $f(x)$ β-computable.

Effectivity: $f : x \mapsto f(x)$ is (α, β)-computable.

Call a representation α of X reducible to α' of X ($\alpha \preceq \alpha'$), if some Turing machine converts α-names into α'-names for *the same* object $x \in X$. Equivalent, i.e., pairwise reducible representations lead to identical notions of uniform computability. It holds, for example,

$$\rho \preceq \rho_<, \ \rho \preceq \rho_>, \quad \rho_< \not\preceq \rho, \ \rho_> \not\preceq \rho, \quad \rho \equiv (\rho_< + \rho_>)$$

We mention two ways of constructing new representations from given ones, details can be found in [15]:

- Let $\alpha_1, \ldots, \alpha_n$ be representations of $X_1, \ldots, X_n$, respectively. There exists a canonical representation, denoted $\prod_{i=1}^n \alpha_i$, of $\prod_{i=1}^n X_i$.
- Let α and β be admissible representations of X and Y, respectively. There exists a canonical representation of $C(X, Y)$, the space of continuous functions $f : X \to Y$, which will be denoted by $[\alpha \to \beta]$.

3 Computability of Sets

For subsets of $\mathbb{R}^n$, several notions of computability are treated in literature. For instance, $\emptyset \neq A \subseteq \mathbb{R}^n$ is Turing-located [6] iff the Euclidean distance function

$$d_A : \mathbb{R}^n \to \mathbb{R}, \qquad x \mapsto \inf_{a\in A} \sqrt{\sum\nolimits_{i=1}^{n} (x_i - a_i)^2}$$

is (ρ^n, ρ)-computable. This d_A can be regarded as continuous analogue to the characteristic function of A which, except for trivial cases $A = \emptyset$ and $A = \mathbb{R}^n$, is never computable nor continuous.

It is easy to see that d_A characterizes the set A uniquely, *provided* A is closed. In other words, we have a representation (from now on called ψ^n) of the hyperspace $\mathfrak{A}^n$ of closed non-empty subsets of $\mathbb{R}^n$: $\bar{\sigma}$ is a ψ^n-name for A iff $\bar{\sigma}$ is a $[\rho^n \to \rho]$-name for d_A. On the other hand, no representation of *all* subsets can exist as there are too many of them: $|2^{\mathbb{R}}| > \mathfrak{c}$.

By definition, $A \in \mathfrak{A}^n$ is Turing-located iff it is ψ^n-computable. Two weaker notions, $\psi^n_<$ and $\psi^n_>$, ask for $[\rho^n \to \rho_>]$- and $[\rho^n \to \rho_<]$-computability of d_A, respectively. Notice the reversed index: $\psi^n_> = [\rho^n \to \rho_<]$.

For compact, i.e. bounded closed, sets it turns out that many problems become computable only if, in addition to the ψ^n-information on A, some *explicit* upper bound on its diameter is supplied. Let

$$\mathfrak{K}^n := \{(A, r) : A \in \mathfrak{A}^n, r \in \mathbb{N}, A \subset [-r, r]^n\} \subset \mathfrak{A}^n \times \mathbb{N}$$

denote the hyperspace of all non-empty compact subsets and $\kappa^n := (\psi^n \times \text{bin})$ its canonical representation; similarly $\kappa^n_<$ and $\kappa^n_>$. We tacitly assume that $\mathfrak{K}^n$ be embedded into $\mathfrak{A}^n$ the obvious way. *Non*-uniform it holds that compact $A \subset \mathbb{R}^n$ is ψ^n-computable iff it is κ^n-computable. Also, obviously $\kappa^n \preceq \psi^n\big|^{\mathfrak{K}^n}$; but $\psi^n\big|^{\mathfrak{K}^n} \not\preceq \kappa^n$.
It is well-known that κ^n (but not ψ^n) permits a computable version of the famous HEINE-BOREL Theorem, see [15]. Even more, the maximum of arbitrary continuous functions over compact sets can effectively be computed:

Lemma 1 (Corollary 6.2.5 in [15]) *Maximum of continuous functions over a compact set* $C(\mathbb{R}^n, \mathbb{R}) \times \mathfrak{K}^n \ni (f, K) \mapsto \max f[K] \in \mathbb{R}$ *is* $\big([\rho^n \to \rho] \times \kappa^n, \rho\big)$*-computable.*

On the other hand, compact sets are quite often described not by κ^n-names but, for example a convex body, as solutions to a system of linear inequalities:

$$L(A, b) = \{x \in \mathbb{R}^n : A \cdot x \le b\}, \quad A \in \mathbb{R}^{m \times n}, \ b \in \mathbb{R}^m$$

The very important question is thus: *Can* $(\rho^{m \times n} \times \rho^m)$*-names for* (A, b) *be converted to* κ^n*-names for* $L(A, b)$*?*
In general, they can't. Even more, LINEAR OPTIMIZATION with degenerate constraint hyperplane configurations is in general not computable! This folkloristic observation, usually avoided by *general position* presumptions, now follows rigorously in the sound framework of Recursive Analysis from its Main Theorem and the following

Example 2 (Discontinuous LP Optimum) *In dimension 2, fix* $f : (x, y) \mapsto x$ *and* $0 \le \varepsilon < 1$*; consider this LP:*

$$\begin{array}{lcl} y & \ge & \varepsilon x \\ y & \le & 0 \\ x & \ge & -1 \\ x & \le & 1 \end{array} \qquad \textit{i.e.} \qquad A = \begin{pmatrix} \varepsilon & -1 \\ 0 & 1 \\ -1 & 0 \\ 1 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

$\varepsilon > 0$**:** $L(A, b) = \text{convHull}\{(0, 0), (-1, 0), (-1, -\varepsilon)\}$, $f[L(A, b)] = [-1, 0]$, $\max f[L(A, b)] = 0$

$\varepsilon = 0$**:** $L(A, b) = [-1, 1] \times \{0\}$, $\max f[L(A, b)] = 1$ □

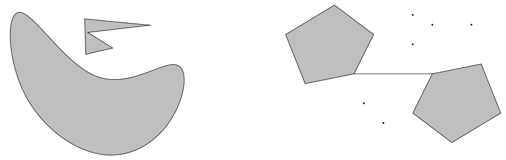
Another problem arises from the fact that the hyperspace of compact subsets of $\mathbb{R}^n$ does not contain halfspaces nor is it effectively closed under intersection or pre-image [3]:

- Binary union $\cup : \mathfrak{A}^n \times \mathfrak{A}^n \to \mathfrak{A}^n$, $(A, B) \mapsto A \cup B$ is $(\psi^n \times \psi^n, \psi^n)$- and $(\kappa^n \times \kappa^n, \kappa^n)$-computable;
- whereas intersection $(A, B) \mapsto A \cap B$ is not even $(\kappa^n \times \kappa^n, \psi^n_<)$-continuous.
- Function pre-image $A \mapsto f^{-1}[A]$ is not $(\kappa^m, \psi^n_<)$-computable, even for computable $f : \mathbb{R}^n \to \mathbb{R}^m$.

4 Computable Solids

It is important to observe that the two-dimensional Example 2 heavily relies on the set $L(A, b)$ to become one-dimensional for $\varepsilon = 0$. In fact, we track down the (non-) computability of LINEAR OPTIMIZATION to an issue of *full-dimensionality* rather than non-degeneracy. More precisely, our new Theorem 4 shows that operations union, intersection, and pre-image become effective when restricting to everywhere full-dimensional sets.
Call set A everywhere full-dimensional (or regular) if it coincides with the closure of its topological interior $\overline{A^\circ}$. This notion generalizes the full-dimensionality from convex sets, cf. Lemma 11 of the appendix. A solid is a regular non-empty proper subset of $\mathbb{R}^n$. The below figure exemplifies, in dimension 2,

A) A SOLID AND B) A NON-SOLID SET.



Let $\mathfrak{X}^n$ denote the hyperspace of all solids in $\mathbb{R}^n$. It holds:

$$A \in \mathfrak{X}^n \quad \underset{d)+h)}{\overset{L.\ 9}{\Longleftrightarrow}} \quad \emptyset \ne A = \overline{\mathbb{R}^n \setminus B} \quad \text{for } \emptyset \ne B := \overline{\mathbb{R}^n \setminus A}$$

Encode $A \in \mathfrak{X}$ simultaneously by $\psi^n_>$-names for both A and B, i.e., effective approximations from below of *two* distances: to the set and to its closed complement. Let η^n denote the corresponding representation.
On the other hand by virtue of Lemma 10d):

$$A \in \mathfrak{X}^n \quad \Longleftrightarrow \quad A = \overline{\mathbb{R}^n \setminus B} \quad \text{for } \textit{some } B \in \mathfrak{A}^n$$

One may thus extend the domain of η^n and encode $A \in \mathfrak{X}^n$ by a $\psi^n_>$-name for A and a $\psi^n_>$-name for *any* B with

$A = \overline{\mathbb{R}^n \setminus B}$. Let ξ^n denote this joint representation. We made the important observation that both approaches are only logically but not computationally equivalent:

Lemma 3 $\eta^n \preceq \xi^n$, *but* $\xi^n \not\preceq \eta^n$. □

It turns out that ξ^n serves better for generating κ^n-names of $L(A,b)$: Many so-far discontinuous operators on closed sets become ξ^n-computable after restricting them to solids. For intersection and pre-image, precautions have to be taken to avoid non-solid results:

Theorem 4 *a) Union is* $(\xi^n \times \xi^n, \xi^n)$*-computable.*

b) Intersection of solids set is $(\xi^n \times \xi^n, \xi^n)$*-computable whenever well-defined, i.e., on* $\{(A,B) : A, B \in \mathfrak{X}^n, A \cap B \in \mathfrak{X}^n\} \subset \mathfrak{X}^n \times \mathfrak{X}^n$

c) Pre-image under continuous open *mappings* $f \in C(\mathbb{R}^n, \mathbb{R}^m)$ *is* $([\rho^n \to \rho^m] \times \xi^m, \xi^n)$*-computable.*

d) $\xi^n \preceq \psi^n$*; even restricted to solids,* $\psi^n|^{\mathfrak{X}^n} \not\preceq \xi^n$.

e) Restricted to non-empty, compact and connected sets, $\psi^n|^{\{A \in \mathfrak{K}^n \text{ connected}\}} \preceq \kappa^n$ □

Combining parts d) and e) and Lemma 3 with Lemma 1, it immediately follows that *any* computable function can effectively be maximized over a body (compact, connected, non-empty, everywhere full-dimensional) $B \subset \mathbb{R}^n$, given by either a ξ^n-name, a η^n-name, a ψ^n-name, or a κ^n-name!

Let us now show how such a name for full-dimensional $L(A,b)$ can be obtained from a $(\rho^{m\times n} \times \rho^m)$-name of (A,b): For $\lambda_0 \in \mathbb{R}$, $0 \neq \lambda \in \mathbb{R}^n$, $H_{\lambda_0,\lambda}$ denotes the closed halfspace $\{x \in \mathbb{R}^n : \sum_{i=1}^n \lambda_i x_i \geq \lambda_0\}$.

Theorem 5 *a) The mapping* $\mathbb{R} \times (\mathbb{R}^n \setminus \{0\}) \ni (\lambda_0, \lambda) \mapsto H_{\lambda_0,\lambda} \in \mathfrak{X}^n$ *is* $(\rho \times \rho^n, \xi^n)$*-computable.*

b) Upon $(\rho^{m\times n} \times \rho^m)$*-input of* $(A,b) \in \mathbb{R}^{m\times n} \times \mathbb{R}^m$, $L(A,b)$ *is* ξ^n*-computable,* provided A *contains no zero-rows and* $L(A,b)$ *is full-dimensional.* □

Let us remark that, apart from full-dimensionality (Example 2), boundedness is an essential prerequisite, too, in order to ensure computability of LINEAR OPTIMIZATION:

Example 6 *In dimension 2, fix* $f : (x,y) \mapsto x$. *The LP*

$$x \leq \varepsilon y \qquad i.e. \qquad A = (1, -\varepsilon), \quad b = (0)$$

defines, for each $0 \leq \varepsilon < 1$, *a full-dimensional domain* $L(A,b)$. *But* $\max f[L(A,b)]$ *is discontinuous in* ε*:* $\max f[L(A,b)] = \infty$ *for* $\varepsilon > 0$, $= 0$ *for* $\varepsilon = 0$. □

5 Conclusion

We presented Turing-computability results for BODIES, i.e., non-empty bounded connected regular closed subsets of $\mathbb{R}^n$. As objects cannot be physically distinguished anyway if they differ only on a nowhere dense set, the full-dimensionality requirement is no restriction in practice. Theorem 5 generalizes previous considerations [4] on computability of linear equalities to *in*equalities. Proof techniques however are entirely different: Topology rather than Linear Algebra, see Appendix A and B.

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A Proofs

Here come the proofs which had so far been omitted due to space limitations. Relying on previous results on set-computability in [3, 15], we will need one additional representation namely of the hyperspace $\mathfrak{O}^n$ of *open* (possibly empty) subsets $U \subseteq \mathbb{R}^n$: A $\theta^n_<$-name for $U \in \mathfrak{O}^n$ is a sequence

$$(x_i, r_i) \in \mathbb{Q}^n \times \mathbb{Q}_+, \quad i \in \mathbb{N}, \quad \text{s.t. } U = \bigcup_{i \in \mathbb{N}} B(x_i, r_i)$$

Similarly, a $\theta^n_>$-name for U is a list of all (!) rational (x, r) such that $B(x, r) \cap (\mathbb{R}^n \setminus U) \neq \emptyset$. Let us point out that representations $\psi^n_>$, $\psi^n_<$, ψ^n, $\kappa^n_<$, $\kappa^n_>$, κ^n can easily be extended to encode also *empty* sets [3]; the trick is to permit ∞ as possible value of the distance function d_A, i.e., consider $[\rho^n \to \bar{\rho}]$-names where $\bar{\rho}$ is a canonical extension of ρ to $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.

Let us now recall some well-known computability results:

Lemma 7 (Toolbox)

a) *Complement* $\mathfrak{O}^n \ni U \mapsto \mathbb{R}^n \setminus U \in \mathfrak{A}^n$ *is both* $(\psi^n_<, \theta^n_>)$*- and* $(\psi^n_>, \theta^n_<)$*-computable.*

b) *Complement* $\mathfrak{A}^n \ni A \mapsto \mathbb{R}^n \setminus A \in \mathfrak{O}^n$ *is both* $(\theta^n_<, \psi^n_>)$*- and* $(\theta^n_>, \psi^n_>)$*-computable.*

c) *Union* $\mathfrak{A}^n \times \mathfrak{A}^n \ni (A, A') \mapsto A \cup A' \in \mathfrak{A}^n$ *is both* $(\psi^n_> \times \psi^n_>, \psi^n_>)$*- and* $(\psi^n_< \times \psi^n_<, \psi^n_<)$*-computable.*

d) *Intersection* $\mathfrak{A}^n \times \mathfrak{A}^n \ni (A, A') \mapsto A \cap A' \in \mathfrak{A}^n$ *is* $(\psi^n_> \times \psi^n_>, \psi^n_>)$*- and* $(\kappa^n_> \times \psi^n_>, \kappa^n_>)$*-computable.*

e) *Pre-image*

$$C(\mathbb{R}^n, \mathbb{R}^m) \times \mathfrak{A}^m \ni (f, A) \mapsto f^{-1}[A] \in \mathfrak{A}^n$$

is $\big([\rho^n \to \rho^m] \times \psi^m_>, \psi^n_>\big)$*-computable.*

f) *Pre-image*

$$C(\mathbb{R}^n, \mathbb{R}^m) \times \mathfrak{O}^m \ni (f, A) \mapsto f^{-1}[A] \in \mathfrak{O}^n$$

is $\big([\rho^n \to \rho^m] \times \theta^m_<, \theta^n_<\big)$*-computable.*

g) *Closure* $\mathfrak{O}^n \ni U \mapsto \overline{U} \in \mathfrak{A}^n$ *is* $(\theta^n_<, \psi^n_<)$*-computable.*

h) *Among compact sets, the property of being empty is* $\kappa^n_>$*-r.e., in other words: Upon* $\kappa^n_>$*-input of compact* $K \in \mathfrak{K}^n$*, a Turing machine can in finite time verify (but not decide) whether* $K = \emptyset$*.*

i) *Type-conversion of functions is effective, that is, we have* UTM*- and* SMN*-like properties: Let* α, β, γ *be admissible representations for* X, Y*, and* Z*, respectively. Then*

$$X \times C(X \times Y, Z) \ni (x, f) \mapsto f(x, \cdot) \in C(Y, Z)$$

is $\big(\alpha \times [\alpha \times \beta \to \gamma], [\beta \to \gamma]\big)$*-computable.*

Proof: See Theorem 5.1.13, Exercise 5.1.21, Theorem 6.2.4, Theorem 3.3.15 in [15]. □

In some of the following proofs, we shall deliberately refer to facts from Topology which are collected in Appendix B.

Proof: [Lemma 3] The first claim is trivial, because any η^n-name for $A \in \mathfrak{X}^n$ is a ξ^n-name for A as well. We will now show that η^n-names for solids cannot in general depend continuously on their respective ξ^n-names. Let us consider the one-dimensional case:

$$U := (-1, -\varepsilon) \cup (+\varepsilon, +1), \quad A := \overline{U}, \quad 0 \leq \varepsilon < 1$$

$B := \mathbb{R} \setminus U$ has $\overline{\mathbb{R} \setminus B} = A$ and $\psi_>$-names for A and B can be chosen which depends continuously on ε. So we have, for each $\varepsilon \geq 0$, a ξ-name for A. But a η-name must contain $\psi_>$-information about $B' := \overline{\mathbb{R} \setminus A}$ rather than B; and for $\varepsilon > 0$,

$$B' = B = (-\infty, -1] \cup [-\varepsilon, +\varepsilon] \cup [+1, +\infty)$$

whereas for $\varepsilon = 0$, $B' = (-\infty, -1] \cup [+1, +\infty)$: The function $\varepsilon \mapsto d_{B'}(0)$ 'jumps' from 0 to 1 as $\varepsilon \searrow 0$ and thus cannot be (lower semi-) continuous. □

Proof: [Theorem 4a)] Let $A, A' \in \mathfrak{X}^n$, $A = \overline{\mathbb{R}^n \setminus B}$ and $A' = \overline{\mathbb{R}^n \setminus B'}$ for $B, B' \in \mathfrak{A}^n$. Keep in mind that a ξ^n-name for A consists of $\psi^n_>$-information about both A *and* B; analogously for A', B'.
By virtue of Lemma 7c), we can effectively obtain $\psi^n_>$-information on $A'' = A \cup A'$. Let us also compute $\psi^n_>$-information on $B'' := B \cap B'$ according to Lemma 7d). Then, $\overline{\mathbb{R}^n \setminus B''} = \overline{(\mathbb{R}^n \setminus B) \cup (\mathbb{R}^n \setminus B')} \overset{\text{L.9g)}}{=} \overline{\mathbb{R}^n \setminus B} \cup \overline{\mathbb{R}^n \setminus B'} = A \cup A'$. We have thus computed $\psi^n_>$-information for $A'' = A \cup A'$ and for *some* closed B'' with $\overline{\mathbb{R}^n \setminus B''} = A''$ which, according to the definition of ξ^n, is sufficient. This also implies regularity, i.e., $A'' \in \mathfrak{X}^n$. □

Other than union, the intersection of regular closed sets is in general *not* regular again, consider $A = [-1,0]$ and $A' = [0,1]$. The requirement that $A \cap A' \in \mathfrak{X}^n$ is therefore necessary for its ξ^n-computability. We will now show, that it is even sufficient:

Proof: [Theorem 4b)] Let $A, A', A'' \in \mathfrak{X}^n$, $A'' = A \cap A'$. By definition, $A = \overline{\mathbb{R}^n \setminus B}$, $A' = \overline{\mathbb{R}^n \setminus B'}$ and $A'' = \overline{\mathbb{R}^n \setminus B''}$ for closed sets B, B', B''.
A $\psi^n_>$-name for A'' is easily obtained from those of A and A' by virtue of Lemma 7d). It is in general *not* possible to compute a $\psi^n_>$-name for B''. On the other hand, $\psi^n_>$-information about *any* closed C with $A'' = \overline{\mathbb{R}^n \setminus C}$ suffices. And $C := B \cup B'$ is easily computable, see Lemma 7c). It thus remains to show $\overline{\mathbb{R}^n \setminus C} = A''$.
To this end, let us apply Lemma 12b). Set $W := \mathbb{R}^n \setminus B''$, $U_1 := \mathbb{R}^n \setminus B$, $U_2 := \mathbb{R}^n \setminus B'$. By presumption, $\overline{W} = A'' = A \cap A' = \overline{U_1} \cap \overline{U_2}$, hence

$$A'' = \overline{W} \overset{\text{L.12}}{=} \overline{U_1 \cap U_2} = \overline{\mathbb{R}^n \setminus C}$$

according to de Morgan's law. □

Also for pre-image, the requirement of f being *open* is essential:

Example 8 (non-open pre-image) *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, $(x,y) \mapsto (x,0)$ and $A_\varepsilon := ([-1,0]\times[-1,+1]) \cup ([0,1]\times[\varepsilon,1])$. This f is continuous but not open; A_ε a connected solid for any $0 \le \varepsilon \le 1$. It is easy to see that ξ^n-information can be presented in such a way that it depends continuously on ε. Also, $B_\varepsilon := f^{-1}[A_\varepsilon]$ is a connected solid for any ε:*

$$B_0 = [-1,+1]\times\mathbb{R}, \qquad B_\varepsilon = [-1,0]\times\mathbb{R}, \ \varepsilon > 0$$

However, ξ^n-information about B_ε cannot depend continuously on ϵ and therefore not on A_ε, either. □

Proof: [Theorem 4c)] Let $A \in \mathfrak{X}^m$, $A = \overline{\mathbb{R}^m \setminus B}$ Use Lemma 7e) to $\psi^n_>$-compute $A' = f^{-1}[A] \subseteq \mathbb{R}^n$ and $B' := f^{-1}[B] \subseteq \mathbb{R}^n$. Then,

$$\overline{\mathbb{R}^n \setminus B'} = \overline{f^{-1}[\mathbb{R}^m \setminus B]} \underset{a)+b)}{\overset{\text{L.13}}{=}} f^{-1}\left[\overline{\mathbb{R}^m \setminus B}\right] = A'$$

as f is both continuous and open. □

Proof: [Theorem 4d)] Let A and B be given by their respective $\psi^n_>$-information. Since ψ^n is equivalent to the join of $\psi^n_<$ and $\psi^n_>$, we need $\psi^n_<$-information about $A = \overline{\mathbb{R}^n \setminus B}$. Employ Lemma 7b) and g) to compute it. □

Proof: [Theorem 4e)] Let connected $A \in \mathfrak{K}^n$ be given by a ψ^n-name. Determine, according to Exercise 5.1.13b in [15], some $x \in A \neq \emptyset$. The mapping $r \mapsto \partial B(x,r)$ is obviously (bin, κ^n)-computable. By virtue of Lemma 7d) and h), we can verify, for each $r \in \mathbb{N}$, whether $A \cap \partial B(x,r) = \emptyset$. As A is compact, such an r exists; and using dove-tailing, we can effectively find one. Now $r + \|x\|_2$ is a bound for A, because A is connected to x. □

Proof: [Theorem 5] For $\lambda_0 \in \mathbb{R}$ and nonzero $\lambda \in \mathbb{R}^n$, consider the affine function

$$f : \mathbb{R}^n \to \mathbb{R}, \qquad x \mapsto \sum\nolimits_{i=1}^{n} \lambda_i x_i - \lambda_0 \ .$$

By Lemma 7i), f is $[\rho^n \to \rho]$-computable upon $(\rho \times \rho^n)$-input of (λ_0, λ). Also in finite dimension, a linear function is open iff it is surjective. As $\lambda \neq 0$, this is the case for f. And the constant regular set $[0,\infty)$ is trivially ξ-computable. By virtue of Theorem 4c), $H_{\lambda_0,\lambda} = f^{-1}\big[[0,\infty)\big]$ is ξ^n-computable.
For part b), the affine function

$$g : \mathbb{R}^n \to \mathbb{R}^m, \qquad x \mapsto \sum\nolimits_{i=1}^{n} A \cdot x - b$$

is *not* open for $m > n$. Instead, let $a_i \in \mathbb{R}^n$ denote the i-th row of A, $i = 1 \dots m$. Since $a_i \neq 0$, all halfspaces $H_i = H_{b_i,a_i}$ are ξ^n-computable according to a). As long as the results are regular, intersection $L(A,b) = \bigcap_{i=1}^m H_i$ is ξ^n-computable: Theorem 4b). □

B Topology

This section contains some topological lemmas which serve as 'tools' for the above proofs. From the many equivalent categorial approaches to set theoretic Topology, we prefer the following one:
A topological space is a set X together with a collection $\mathfrak{O} \supseteq \{\emptyset, X\}$ of subsets of X, closed under finite intersections and arbitrary unions. Members $U \in \mathfrak{O}$ will be called

open, complements $X \setminus U$ are closed. For $M \subseteq X$, let

$$\overline{M} := \bigcap_{\substack{M \subseteq A \subseteq X \\ A \text{ closed}}} A \qquad \text{and} \qquad \overset{\circ}{M} := \bigcup_{\substack{U \subseteq M \\ U \text{ open}}} U$$

denote its closure and interior, respectively. The following facts are straight forward to verify:

Lemma 9 *Let* I *denote an arbitray index set and* $G, H, M_i \subseteq X$, $i \in I$.

a) $\bigcup_{i \in I} \overset{\circ}{M_i} \subseteq \left(\bigcup_{i \in I} M_i\right)^\circ$ *e)* $\bigcap_{i \in I} \overset{\circ}{M_i} \supseteq \left(\bigcap_{i \in I} M_i\right)^\circ$

b) $\bigcap \overline{M_i} \supseteq \overline{\bigcap M_i}$ *f)* $\bigcup \overline{M_i} \subseteq \overline{\bigcup M_i}$

c) $G^\circ \cap H^\circ = (G \cap H)^\circ$ *g)* $\overline{G} \cup \overline{H} = \overline{G \cup H}$

d) $(X \setminus G)^\circ = X \setminus \overline{G}$ *h)* $\overline{X \setminus H} = X \setminus H^\circ$

Lemma 10 *Let* U, V, W *and* A, B, C *denote, respectively, open and closed subsets of* X.

a) $U \subseteq \overset{\circ}{\overline{U}}$ *c)* $U = A^\circ \quad \Rightarrow \quad \overset{\circ}{\overline{U}} = U$

b) $A \supseteq \overline{A^\circ}$ *d)* $A = \overline{U} \quad \Rightarrow \quad \overline{A^\circ} = A$

Lemma 11 *Let* $C \subseteq \mathbb{R}^n$ *be non-empty, closed, and convex. The following are equivalent:*

a) $\dim(C) = n$ *b)* $C^\circ \neq \emptyset$ *c)* $\overline{C^\circ} = C$

Proof: **c)⇒b)** $\emptyset \neq C \overset{c)}{=} \overline{C^\circ}$, thus $\emptyset \neq C^\circ$.

b)⇒c) Fix $p \in C^\circ$ and $r > 0$ such that $B(p, r) \subseteq C$. Let $x \in C$, we show $x \in \overline{C^\circ}$: This proves $C \subseteq \overline{C^\circ}$, the other inclusion follows from Lemma 10b).
W.l.o.g. $x = 0$, otherwise consider $C' := -x + C$. By convexity, $x_n := p/n \in C$. Even more, $B(x_n, r/n) \subseteq C$. Thus $x_n \in C^\circ$ for all $n \in \mathbb{N}$, hence $x = \lim x_n \in \overline{C^\circ}$.

b)⇒a) Let $B(p, r) \subseteq C \subseteq \mathbb{R}^n$. Then by monotonicity,

$$n = \dim B(p, r) \leq \dim C \leq \dim \mathbb{R}^n = n \ .$$

a)⇒b) By presumption, the affine hull of C spans whole $\mathbb{R}^n$. W.l.o.g. $0 \in C$, i.e., one may consider the linear rather than affine hull. Choose a basis $b_1, \ldots, b_n \in C$ for $\mathbb{R}^n$ and let $b_0 := 0 \in C$. By convexity, the simplex S spanned by $b_0, b_1, \ldots, b_n$ is contained in C. Based on the indepence of the b_i, it is easy to see that already S has positive volume and non-empty interior. □

Lemma 12 *a) Let* $U \subseteq X$ *be open,* $Q \subseteq X$ *arbitrary but dense in* X. *Then* $\overline{U \cap Q} = \overline{U}$.

b) Let $U_1, U_2, W \subseteq X$ *open and* $\overline{U_1} \cap \overline{U_2} = \overline{W}$. *Then* $\overline{W} = \overline{U_1 \cap U_2}$.

c) Let X *be a Baire space,* $U_n, W \subseteq X$ *open for* $n \in \mathbb{N}$ *and* $\bigcap_{n \in \mathbb{N}} \overline{U_n} = \overline{W}$. *Then* $\overline{W} = \overline{\bigcap_n U_n}$.

Proof:

a) $U \cap Q \subseteq U$ yields $\overline{U \cap Q} \subseteq \overline{U}$ by monotonicity. For the reversed inequality let $A := \overline{U \cap Q}$. The closed set $A' := A \cup (X \setminus U)$ contains $(Q \cap U) \cup (Q \setminus U) = Q$ so, again by monotonicity, $A' = \overline{A'} \supseteq \overline{Q} = X$. But $A' = X$ implies $U \subseteq A = \overline{U \cap Q}$.

b) Let us first treat the case $W = X$: Setting $U := U_1$ and $Q := U_2$, part a) gives the desired claim which is, in words: The intersection of two open dense subsets is dense again.

For $W \subsetneq X$, let $U_i' := U_i \cup (X \setminus \overline{W})$: These sets are open and dense in X according to Lemma 9g); hence $U_1' \cap U_2'$ is, too: the case just treated. One may thus let $Q := U_1' \cap U_2'$ and apply a) once again to obtain $\overline{W \cap Q} = \overline{W}$. Verifying $W \cap Q = W \cap U_1 \cap U_2$, it follows $\overline{U_1 \cap U_2} \supseteq \overline{W}$; the other inclusion holds by virtue of Lemma 9b).

c) Recall that in a Baire space by definition, the countable intersection of open dense subsets is dense again. This proves the case $W = X$ from which the case $W \subsetneq X$ follows as above. □

Lemma 13 *Let* X, Y *be topological spaces,* $f : X \to Y$, $G \subseteq X$, $H \subseteq Y$.

a) If f *is continuous, then*

$$f\left[\overline{G}\right] \subseteq \overline{f[G]} \qquad f^{-1}\left[\overline{H}\right] \supseteq \overline{f^{-1}[H]} \qquad f^{-1}\left[\overset{\circ}{H}\right] \subseteq \left(f^{-1}[H]\right)^\circ$$

b) If f *is open, then*

$$f\left[\overset{\circ}{G}\right] \subseteq \left(f[G]\right)^\circ \qquad f^{-1}\left[\overline{H}\right] \subseteq \overline{f^{-1}[H]} \qquad f^{-1}\left[\overset{\circ}{H}\right] \supseteq \left(f^{-1}[H]\right)^\circ$$

c) If f *is closed, then* $\quad f\left[\overline{G}\right] \supseteq \overline{f[G]}$.