Random numbers and an incomplete immune recursive set

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Abstract. Generalizing the notion of a recursively enumerable (r.e.) set to sets of real numbers and other metric spaces is an important topic in computable analysis (which is the Turing machine based theory of computable real number functions). A closed subset of a computable metric space is called *r.e. closed*, if all open rational balls which intersect the set can be effectively enumerated and it is called *effectively separable*, if it contains a dense computable sequence. Both notions are closely related and in case of Euclidean space (and complete computable metric spaces in general) they actually coincide. Especially, both notions are generalizations of the classical notion of an r.e. subset of natural numbers. However, in case of incomplete metric spaces these notions are distinct. We use the immune set of random natural numbers to construct a recursive immune "tree" which shows that there exists an r.e. closed subset of some incomplete subspace of Cantor space which is not effectively separable. Finally, we transfer this example to the incomplete space of rational numbers (considered as a subspace of Euclidean space).

Keywords: computable analysis, r.e. closed sets, random numbers, Kolmogorov complexity, immune sets.

1 Introduction

In classical recursion theory the notions of a recursively enumerable (r.e.) and a recursive set play an essential role [6]. One interesting topic of research in computable analysis (which is the theory of real number functions which can be computed by Turing machines) is concerned with the generalization of these notions to Euclidean space and other types of spaces [9]. In the case of computable metric spaces some central notions of effectivity for closed subsets and their mutual relationship can be visualized as shown in Figure 1. The displayed results have been obtained in [3] from a very uniform point of view and each arrow in the diagram does not only indicate an implication but an effective reducibility. Below, we will precisely define the notions which are relevant to the present paper. In case of Euclidean space (and other computable metric spaces

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which roughly speaking have to be "effectively locally compact") the vertical arrows can be reversed and thus the three horizontal layers of the diagram collapse [4]. However, a number of examples have been presented in [3] which prove that some of these notions have to be distinguished in the general case of computable metric spaces.

$$
\begin{tikzcd}
 & \text{recursive} \arrow[dl] \arrow[r] & \text{co-r.e.} \\
\text{r.e.} = \text{upper semi-located} & \text{located} \arrow[l] \arrow[u] \arrow[r] & \text{lower semi-located} \arrow[u] \\
\textit{effectively separable} \arrow[u] & \text{strongly recursive} \arrow[u] \arrow[r] & \text{strongly co-r.e.} \arrow[u]
\end{tikzcd}
$$

Fig. 1. Notions of effectivity for closed subsets of computable metric spaces

In this paper we will deal with another example of this type which has already been announced but not proved in [3] and which shows that there are naturally defined computable metric spaces and examples of closed subsets which fulfill all displayed effectivity notions besides effective separability. The results of [3] imply that such spaces and sets necessarily have to be incomplete (since recursive enumerability implies effective separability in case of complete spaces or sets). Even the space of rational numbers (endowed with subtopology of the Euclidean metric) admits subsets of the mentioned type. A glance at Figure 1 shows that it suffices to show that there exists a strongly recursive closed subset which is not effectively separable to prove the claim.

In the following Section 2 we define some basic concepts from computable analysis and we introduce the relevant notions of effectivity for closed subsets. In Section 3 we use the set of natural random numbers to construct a recursive subset of Cantor space. This "tree" is our basic example of a strongly recursive closed subset (of some incomplete subspace of Cantor space) which is not effectively separable. Actually, we can prove even a stronger result which shows that the constructed tree is immune in the sense that it does not include an infinite computable sequence. In Section 4 we will transfer our example to the space of rational numbers.

2 Preliminaries from computable analysis

In this section we introduce some basic concepts from computable analysis; for details we refer the reader to [9]. We start with computable metric spaces.

Definition 1 (Computable metric space). We will call a triple (X, d, α) a *computable metric space*, if

1. $d : X \times X \to \mathbb{R}$ is a metric on X,
2. $\alpha : \mathbb{N} \to X$ is dense in X, i.e. the closure of range(α) is equal to X,
3. $d \circ (\alpha \times \alpha) : \mathbb{N}^2 \to \mathbb{R}$ is a computable (double) sequence of real numbers.

Here we assume that the reader is familiar with the notion of a computable sequence of real numbers in sense of computable analysis (cf. for instance [7, 5, 10, 9]). We mention some standard examples of computable metric spaces which we will use in the following. Therefore, we first introduce some technical notations. By $\{0,1\}^*$ we denote the set of *finite words* over the alphabet $\{0,1\}$ and by $\mathcal{C} := \{0,1\}^\omega$ we denote the set of infinite binary sequences (and occasionally we assume $\{0,1\}^\omega \subseteq \mathbb{N}^\mathbb{N}$). By wp we denote the concatenation of a word w with a sequence (or a finite word) p, and the *prefix order* will be denoted by "$\sqsubseteq$", i.e. $w \sqsubseteq p$ holds, whenever w is a prefix of p. By 0^ω we denote the zero sequence and by $\mathcal{Z} := \{0,1\}^*0^\omega$ the set of all binary sequences which are eventually zero. Similarly, we denote by $w\{0,1\}^\omega$ the set of sequences with prefix w. By $\langle n,k\rangle := 1/2(n+k)(n+k+1)+k$ we denote the *Cantor pairing* of $n,k \in \mathbb{N}$ which can be extended inductively to arbitrary finite tuples and we define $\widehat{0} := 0$ and $\widehat{n+1} := 1$ for all $n \in \mathbb{N}$.

Example 2 (Computable metric spaces).

1. $(\mathbb{R}, d_\mathbb{R}, \alpha_\mathbb{R})$ with the *Euclidean metric*
$$d_\mathbb{R}(x,y) := |x-y|$$
and some standard enumeration $\alpha_\mathbb{R}\langle i,j,k\rangle := \frac{i-j}{k+1}$ of the set $\mathbb{Q}$ of rational numbers, is a computable metric space.
2. $(\mathbb{Q}, d_\mathbb{Q}, \alpha_\mathbb{Q})$ with the restriction $d_\mathbb{Q}$ of $d_\mathbb{R}$ to $\mathbb{Q}\times\mathbb{Q}$ in the source and the restriction $\alpha_\mathbb{Q}$ of $\alpha_\mathbb{R}$ to $\mathbb{Q}$ in the target, is a computable metric space.
3. $(\{0,1\}^\omega, d_\mathcal{C}, \alpha_\mathcal{C})$ with the *Cantor metric*
$$d_\mathcal{C}(p,q) := \begin{cases} 2^{-\min\{i\in\mathbb{N}\,:\,p(i)\neq q(i)\}} & \text{if } p \neq q \\ 0 & \text{else} \end{cases}$$
and some enumeration $\alpha_\mathcal{C}\langle k, \langle n_1,...,n_k\rangle\rangle := \widehat{n_1}...\widehat{n_k}0^\omega$ of $\mathcal{Z} := \{0,1\}^*0^\omega$ is a computable metric space.
4. $(\mathcal{Z}, d_\mathcal{Z}, \alpha_\mathcal{Z})$ with the restriction $d_\mathcal{Z}$ of $d_\mathcal{C}$ to $\mathcal{Z}\times\mathcal{Z}$ in the source and the restriction $\alpha_\mathcal{Z}$ of $\alpha_\mathcal{C}$ to $\mathcal{Z}$ in the target, is a computable metric space.

The proofs that these spaces are actually computable metric spaces are straightforward (cf. [2, 8, 9]). In the following we will refer to these spaces simply by writing $\mathbb{R}, \mathbb{Q}, \mathcal{C}$, and $\mathcal{Z}$, respectively. The spaces $\mathbb{Q}$ and $\mathcal{Z}$ are typical examples of *incomplete* computable metric spaces and $\mathbb{R}$ and $\mathcal{C}$ are their natural completions, respectively.

In computable analysis a function $f :\subseteq X \to Y$ is called *computable*, if there exists a Turing machine which transfers each infinite sequence $p \in \Sigma^\omega$ (over some alphabet Σ) that represents some input $x \in X$ into some sequence $q \in \Sigma^\omega$ which represents the result $f(x)$. Of course, such a Turing machine has to compute infinitely long, but in the long run each infinite input sequence is transformed into an appropriate output sequence. Here and in the following, the inclusion symbol "$\subseteq$" is used to denote functions which are possibly partial. It is a reasonable restriction that only Turing machines with one-way output tape are allowed (because otherwise the output after some finite time would be useless, since it could be changed by the machine later on). More formally, a *representation* of a set X is a surjective mapping $\delta :\subseteq \Sigma^\omega \to X$. Using this notion we can define computable functions precisely.

Definition 3 (Computable functions). Let δ and δ' be representations of X and Y, respectively. A function $f :\subseteq X \to Y$ is called (δ, δ')*–computable*, if there exists a Turing machine M such that $f\delta(p) = \delta' F_M(p)$ for all $p \in \mathrm{dom}(f\delta)$.

Here, $F_M :\subseteq \Sigma^\omega \to \Sigma^\omega$ denotes the function computed by Turing machine M. The diagram in Figure 2 illustrates the situation.

$$\begin{tikzcd}
\Sigma^\omega \arrow[r, "F_M"] \arrow[d, "\delta"'] & \Sigma^\omega \arrow[d, "\delta'"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Fig. 2. Computability with respect to representations

With each computable metric space (X, d, α) we can canonically associate its *Cauchy representation* δ_X, where $\delta_X(01^{n_0+1}01^{n_1+1}01^{n_2+1}...) = \lim_{i\to\infty} \alpha(n_i)$ for all $n_i \in \mathbb{N}$ such that $d(\alpha(n_i), \alpha(n_j)) \le 2^{-j}$ for all $i > j$. Roughly speaking, $\delta_X(p) = x$, if p encodes a Cauchy sequence in range(α) which rapidly converges to x. Occasionally, we will also use some standard representation $\delta_\mathbb{N}$ of the natural numbers $\mathbb{N} := \{0, 1, 2, ...\}$. In the following we will say for short that a function on $\mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathcal{Z}$ and $\mathcal{C}$ is *computable*, if it is computable with respect to the corresponding representation $\delta_\mathbb{N}, \delta_\mathbb{Q}, \delta_\mathbb{R}, \delta_\mathcal{Z}$ and $\delta_\mathcal{C}$, respectively. The class of functions $f : \Sigma^\omega \to \Sigma^\omega$ over the alphabet $\Sigma := \{0, 1\}$ which are computable by Turing machines coincides with the class of computable functions with respect to $\delta_\mathcal{C}$. Thus, the introduced notions are consistent.

A point $x \in X$ of a recursive metric space (X, d, α) with Cauchy representation δ_X is called *computable*, if there exists some computable p such that $\delta_X(p) = x$. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is called *computable*, if the corresponding function $f : \mathbb{N} \to X$ with $f(n) := x_n$ is $(\delta_\mathbb{N}, \delta_X)$–computable.

We close this section with a definition of those effectivity notions for subsets which will be used throughout this paper. We will use the notation $B(x, \varepsilon) := \{y \in X : d(x, y) < \varepsilon\}$ for the *open ball* with center x and radius ε and analogously we denote by $\overline{B}(x, \varepsilon) := \{y \in X : d(x, y) \le \varepsilon\}$ the corresponding *closed ball*. In general, $\overline{A}$ denotes the *topological closure* of a subset $A \subseteq X$. Moreover, we use the abbreviation $\overline{n} := \alpha_\mathbb{R}(n)$ for the rational numbered by $n \in \mathbb{N}$.

Definition 4 (Recursively enumerable closed subsets). Let (X, d, α) be a computable metric space and let $A \subseteq X$ be a closed subset.

1. A is called *r.e. closed*, if the set $\{\langle n, k\rangle \in \mathbb{N} : A \cap B(\alpha(n), \overline{k}) \neq \emptyset\}$ is r.e.
2. A is called *strongly co-r.e. closed*, if the set $\{\langle n, k\rangle \in \mathbb{N} : A \cap \overline{B}(\alpha(n), \overline{k}) = \emptyset\}$ is r.e.
3. A is called *strongly recursive closed*, if A is r.e., as well as strongly co-r.e. closed.

Our aim is to construct a strongly recursive closed subset which is not effectively separable. Therefore we define effective separability.

Definition 5 (Effective separability). Let (X, d, α) be a computable metric space and let $A \subseteq X$ be a subset. Then A is called *effectively separable*, if there exists a computable sequence $f : \mathbb{N} \to X$ such that $\mathrm{range}(f) = \{f(n) : n \in \mathbb{N}\}$ is dense in A.

3 An immune recursive tree

In this section we construct a subset $T \subseteq \{0,1\}^\omega$ with some interesting properties. Using a non-standard but intuitive terminology, we will call such subsets *trees* for short (even if T is not closed). The tree T that we will construct is recursive in the sense that we can effectively decide which nodes belong to a path of the tree. Especially, T is a strongly recursive closed subset of $\mathcal{Z} = \{0,1\}^*0^\omega$. Since $T \subseteq \mathcal{Z}$, all paths of T are computable, but T has the property that there is no algorithm which, given a node of T as input, finds some path in T which goes through the given node. This implies that T is not effectively separable. And more than this, we will even prove that T does not contain an infinite computable sequence.

For the construction of T we will use some notions from recursion theory [6]. Let $\varphi : \mathbb{N} \to P$ denote some *admissible Gödel numbering* of the set of partial computable functions $P := \{f :\subseteq \mathbb{N} \to \mathbb{N} : f \text{ computable}\}$. Then the *Kolmogorov complexity* of a number $n \in \mathbb{N}$ is defined by $K(n) := \min\{i \in \mathbb{N} : \varphi_i(0) = n\}$, which is the "shortest program" that can produce n. Without loss of generality, we can assume that $K(\varphi_i \circ \varphi_j(n))$ is bounded by a term linear in i, j, n, i.e. we can assume that there exists some constant $c \in \mathbb{N}$ such that for all $i, j, n \in \mathbb{N}$ with $n \in \mathrm{dom}(\varphi_i \circ \varphi_j)$

$$K(\varphi_i \circ \varphi_j(n)) \leq c(i+1)(j+1)(n+1) + c.$$

By $L := \{n \in \mathbb{N} : K(n) < n\}$ we denote the set of *nonrandom* or *lawful numbers*. Intuitively, a number $n \in \mathbb{N}$ is *random*, i.e. $n \notin L$, if n is its own shortest description (thus there is no algorithm i, smaller than n itself, which produces $\varphi_i(0) = n$). Without loss of generality, we assume that $0, 1 \in \mathbb{N} \setminus L$ are random numbers. It is known that the set of nonrandom numbers L is *simple*, i.e. it is r.e. and its complement is infinite but does not contain any infinite r.e. subset. Thus, the set of random numbers $\mathbb{N} \setminus L$ is *immune*: no algorithm can produce more than finitely many random numbers (cf. [6] for the definition and properties of random numbers). Now we use the set L of nonrandom numbers to construct a tree T.

Definition 6 (The tree T). Let $s : \mathbb{N} \to \mathbb{N}$ be some computable function which enumerates the set of nonrandom numbers, i.e. $\text{range}(s) = L$ and let $t :\subseteq \mathbb{N} \to \mathbb{N}$ be defined by $t(k) := \min\{m \in \mathbb{N} : s(m) = k\}$. Now let $T \subseteq \mathcal{Z} = \{0,1\}^*0^\omega$ be the set which contains all sequences

$$p = 1^{n_0+1}0^{t(k_0)+1}1^{n_1+1}0^{t(k_1)+1}1^{n_2+1}0^{t(k_2)+1}...1^{n_{j-1}+1}0^{t(k_{j-1})+1}1^{n_j+1}0^\omega$$

such that $j \in \mathbb{N}$, $n_0, ..., n_j \in \mathbb{N}$, $k_0, ..., k_{j-1} \in L$ and $k_j \in \mathbb{N} \setminus L$ and the equations

$$\begin{cases} k_0 &= n_0 \\ k_{i+1} &= k_i^2 + n_{i+1} \end{cases}$$

hold for all $i = 0, ..., j-1$.

By construction T consists only of sequences which are eventually zero. More than this, T is even closed in $\mathcal{Z}$ but it is not complete and hence not closed in $\{0,1\}^\omega$. The closure $\overline{T}$ of T contains sequences with infinitely many ones.

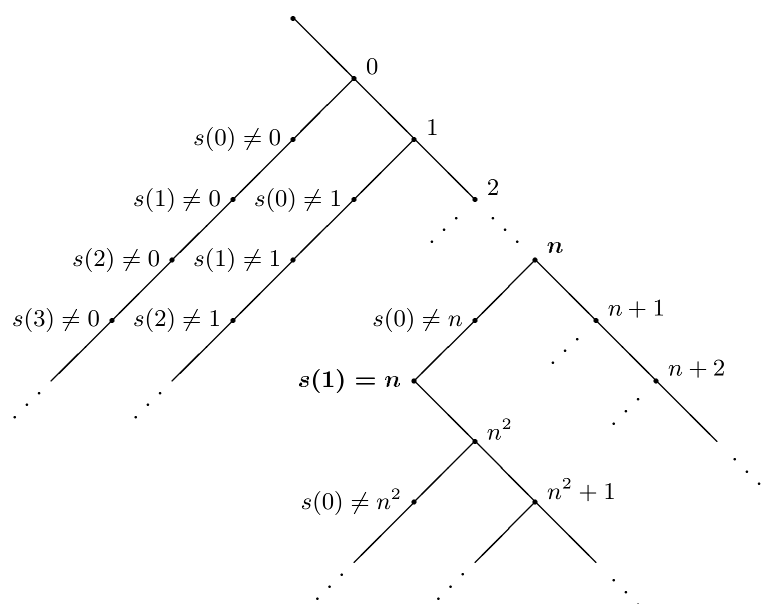

Fig. 3. A part of tree T with a nonrandom number $n = s(1) \in L$

The intuition of the construction of T is as follows (cf. Figure 3). Let us consider T as a subset of the full binary tree $\{0,1\}^\omega$ and consider the rightmost path of ones in this tree: we enumerate consecutive nodes on this path with natural numbers $0, 1, 2, ...$ and so on. The path p through node n which continues to the left with zeroes, i.e. $p = 1^{n+1}0^\omega$, belongs to T, if and only if n is a random number, i.e. $n \in \mathbb{N} \setminus L$. With any node i of path p we could associate a test which tries to prove that n is nonrandom, i.e. it is checked whether $n{=}s(i)$? If the test fails, then we proceed within the next step and check $n{=}s(i+1)$? If n is random, we follow the path p in this way until infinity. If n is nonrandom, then some test $n = s(i)$ will succeed and the consecutive sequence of zeroes is stopped at this node $i = t(n)$. In this case the path p is continued to the right, i.e. it has $1^{n+1}0^{t(n)+1}1$ as prefix. Again we associate numbers $n^2, n^2{+}1, n^2{+}2, ...$ and so on with the nodes of the following rightmost path and we continue in this subtree as described before. Actually, this intuitive description of the construction of T leads to the following result.

Proposition 7. *The set $P := \{w \in \{0,1\}^* : (\exists p \in T)\ w \sqsubseteq p\}$ is recursive.*

Proof. Let $w \in \{0,1\}^*$ be some word. In order to describe how one can decide whether $w \in P$ holds, we distinguish four cases.
1. Case: $w = 1^{n_0+1}0^{m_0+1}1^{n_1+1}0^{m_1+1}...1^{n_j+1}0^{m_j+1}$, $n_0, ..., n_j, m_0, ..., m_j \in \mathbb{N}$.
We compute $k_0 := n_0$, $k_{i+1} := k_i^2 + n_{i+1}$ for $i = 0, ..., j-1$ and we verify $s(m_i) = k_i$ for $i = 0, ..., j-1$ and $s(m) \neq k_i$ for $m < m_i$ and $i = 0, ..., j$. If one of these conditions is violated, then obviously $w \notin P$. If all conditions are fulfilled, then w is a prefix of some $p \in T$: either $k_j \in \mathbb{N} \setminus L$ is random and hence $w0^\omega \in T$ or otherwise $t(k_j) \geq m_j$ exists and there is some random number among the numbers $k_j^2, k_j^2+1, k_j^2+2, k_j^2+3, ...$, say k_j^2+l, such that $w0^{t(k_j)-m_j}1^{l+1}0^\omega \in T$ in this case.
2. Case: $w = 1^{n_0+1}0^{m_0+1}1^{n_1+1}0^{m_1+1}...1^{n_{j-1}+1}0^{m_{j-1}+1}1^{n_j+1}$ with $n_0, ..., n_j$, $m_0, ..., m_{j-1} \in \mathbb{N}$.
We compute $k_0 := n_0$, $k_{i+1} := k_i^2 + n_{i+1}$ for $i = 0, ..., j-2$ and we verify $s(m_i) = k_i$ and $s(m) \neq k_i$ for $m < m_i$ and $i = 0, ..., j-1$. If one of these conditions is violated, then obviously $w \notin P$. If all conditions are fulfilled, then w is a prefix of some $p \in T$: there is some random number among the numbers $k_{j-1}^2 + n_j, k_{j-1}^2 + n_j + 1, k_{j-1}^2 + n_j + 2, ...$, say $k_{j-1}^2 + n_j + l$, such that $w1^l0^\omega \in T$ in this case.
3. Case: w starts with a zero. In this case obviously $w \notin P$.
4. Case: w is the empty word. In this case obviously $w \in P$. □

Since open balls and closed balls in $\{0,1\}^\omega$ are both of the form $w\{0,1\}^\omega$ with $w \in \{0,1\}^*$ and

$$P = \{w \in \{0,1\}^* : w\{0,1\}^\omega \cap T \neq \emptyset\} = \{w \in \{0,1\}^* : w\{0,1\}^\omega \cap \overline{T} \neq \emptyset\},$$

we obtain the following corollary.

Corollary 8. *T is a strongly recursive closed subset of $\mathcal{Z}$ and its closure $\overline{T}$ is a strongly recursive closed subset of $\{0,1\}^\omega$.*

Although the tree T is recursive there is no computable strategy which determines a path through a given node. In order to prove this result we start with a lemma which shows how the square terms in the definition of the tree T can be used.

Lemma 9. *Let $f :\subseteq \mathbb{N} \to \mathbb{N}$ be a computable function with $f\langle n, 0\rangle \geq n$ for all $n \in \mathbb{N}$ and let $g :\subseteq \mathbb{N} \to \mathbb{N}$ be defined by*

$$\begin{cases} g\langle n, 0\rangle & := f\langle n, 0\rangle \\ g\langle n, k+1\rangle & := (g\langle n, k\rangle)^2 + f\langle n, k+1\rangle \end{cases}$$

Then g is computable and the sequence $g\langle n, 1\rangle, g\langle n, 2\rangle, g\langle n, 3\rangle$... is either finite or all values are defined. In both cases the sequence does contain no random number for almost all fixed values $n \in \mathbb{N}$.

Proof. Obviously, g is a computable function since f is computable. Let $i \in \mathbb{N}$ be such that $\varphi_i = g$. Then there is some constant $c \in \mathbb{N}$ such that

$$K(g\langle n, k\rangle) \leq c(i+1)(n+1)(k+1) + c$$

holds for all $n, k \in \mathbb{N}$ such that $g\langle n, k\rangle$ exists. On the other hand, $g\langle n, k\rangle \geq n^{(2^k)}$ for all values $k \in \mathbb{N}$ such that $g\langle n, k\rangle$ exists. Thus, if n is sufficiently large and itself nonrandom, then $g\langle n, k\rangle > K(g\langle n, k\rangle)$ follows for all $k > 0$ such that $g\langle n, k\rangle$ exists. Thus, for almost all $n \in \mathbb{N}$ the sequence $g\langle n, 1\rangle, g\langle n, 2\rangle, g\langle n, 3\rangle$... does not contain a random value. $\square$

With the help of this lemma we can prove the main result of this section.

Theorem 10. *T is a strongly recursive closed subset of $\mathcal{Z}$ which is not effectively separable.*

Proof. It has already been proved in Corollary 8 that T is strongly recursive closed. It remains to show that T is not effectively separable. Let us assume that there is a computable sequence $(p_i)_{i\in\mathbb{N}}$ in $\{0,1\}^*0^\omega$ such that $\{p_i : i \in \mathbb{N}\}$ is dense in T. Then there exists a computable function $h : \mathbb{N} \to \mathbb{N}$ such that $1^{n+1}0 \sqsubseteq p_{h(n)}$, i.e. h finds a path associated with node n. Let us define a function $f :\subseteq \mathbb{N} \to \mathbb{N}$ by $f\langle n, 0\rangle := n$ and $f\langle n, j+1\rangle = k$, if and only if

$$(\exists n_1, ..., n_j, m_0, ..., m_j \in \mathbb{N})\, 1^{n+1}0^{m_0+1}1^{n_1+1}0^{m_1+1}...1^{n_j+1}0^{m_j+1}1^{k+1}0 \sqsubseteq p_{h(n)}$$

for all $n, j, k \in \mathbb{N}$. Then f is computable too. We mention that $f\langle n, j\rangle$ is undefined for all random numbers $n \in \mathbb{N} \setminus L$ and $j > 0$ and it is defined for all nonrandom numbers $n \in L$ and a certain initial segment $j = 0, ..., \iota$. Then the function $g :\subseteq \mathbb{N} \to \mathbb{N}$, defined according to Lemma 9, is computable as well and there exists some sufficiently large nonrandom $n \in L$ such that the sequence $g\langle n, 0\rangle, g\langle n, 1\rangle, g\langle n, 2\rangle$... does not contain a random number. For this fixed n there exists some ι such that

$$p_{h(n)} = 1^{f\langle n,0\rangle+1}0^{tg\langle n,0\rangle+1}...1^{f\langle n,\iota-1\rangle+1}0^{tg\langle n,\iota-1\rangle+1}1^{f\langle n,\iota\rangle+1}0^\omega \in T$$

and thus $g\langle n, \iota\rangle \in \mathbb{N} \setminus L$ is random by definition of T. But this is a contradiction to the choice of n! $\square$

Now we can use the same method to prove even a stronger result which shows that there is no infinite computable sequence in T at all. In the following we will call a closed subset $A \subseteq X$ of a computable metric space X *immune*, if there exists no computable sequence $f : \mathbb{N} \to X$ such that $\text{range}(f) = \{f(n) : n \in \mathbb{N}\}$ is infinite and included in A. The proof of the following result is a refined version of the previous proof and it is also based on Lemma 9. In the previous proof density allowed to construct a function f starting from the rightmost path of T. Now we will construct a similar function f but starting deeper within the tree. We will use the fact that $\mathbb{N} \setminus L$ is immune in order to prove that any infinite sequence in the tree goes arbitrary deep into the tree.

Theorem 11. *T is an immune set.*

Proof. For each

$$p = 1^{n_0+1}0^{t(k_0)+1}1^{n_1+1}0^{t(k_1)+1}1^{n_2+1}0^{t(k_2)+1}...1^{n_{j-1}+1}0^{t(k_{j-1})+1}1^{n_j+1}0^\omega \in T$$

let $\zeta(p) := k_j := k_{j-1}^2 + n_j$ and $\mu(p) := j$. Thus, $\zeta(p)$ is "the random number" encoded by p and $\mu(p) + 1$ is "the depth" of p, i.e. the number of occurrences of "10" in p. Let us consider $\zeta :\subseteq \{0,1\}^\omega \to \mathbb{N}$ as function. Obviously, ζ with $\text{dom}(\zeta) := \{p : \mu(p) = n\}$ is computable for any fixed $n \in \mathbb{N}$.

Let us assume that $(p_i)_{i\in\mathbb{N}}$ is a computable sequence in $\{0,1\}^\omega$ such that the range $\{p_i : i \in \mathbb{N}\}$ is infinite and included in T.

We prove by induction on n that for all $n \in \mathbb{N}$ there are infinitely many different sequences p_i with $\mu(p_i) \geq n$. This is obvious for $n = 0$. Let us assume that we have proved the claim for n. We will show that the assumption that there are only finitely many p_i with $\mu(p_i) \geq n + 1$ leads to a contradiction. By induction hypothesis and this assumption there are infinitely many p_i with $\mu(p_i) = n$. Since $\{p_i : \mu(p_i) \geq n + 1\}$ is finite, it follows that the set of indices $\{i \in \mathbb{N} : \mu(p_i) < n+1\}$ is r.e. On the other hand, $\{i \in \mathbb{N} : \mu(p_i) \geq n\}$ is r.e. as well and thus $M := \{i \in \mathbb{N} : \mu(p_i) = n\}$ is r.e. too. Moreover, $i \mapsto \zeta(p_i)$ is computable on M and thus $Z := \{\zeta(p_i) : i \in M\}$ is r.e. too. Since $\{p_i : \mu(p_i) = n\}$ is infinite, it follows that the set of numbers $k \in \mathbb{N}$ such that 1^k is a subword of some p_i with $\mu(p_i) = n$, has to be infinite too and since $\zeta(p_i) \geq k - 1$ in this case, it follows that Z is infinite as well. But this is a contradiction since $Z \subseteq \mathbb{N} \setminus L$ and $\mathbb{N} \setminus L$ is immune. Thus, there have to be infinitely many p_i with $\mu(p_i) \geq n + 1$. This finishes the induction.

Since $0, 1 \in \mathbb{N} \setminus L$ are random, we obtain $\mu(p) \geq n \implies \zeta(p) \geq 2^{(2^n)}$ for all $n \geq 1$. Thus, as a consequence of the previous claim, for each $n \in \mathbb{N}$ there are infinitely many p_i with $\zeta(p_i) \geq n$. We can conclude that there exist total computable functions $\gamma, \sigma, h : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$ and $i = \gamma(n)$ there exist $n_0, ..., n_i, m_0, ..., m_{i-1} \in \mathbb{N}$ such that

$$\sigma(n) = s(m_{i-1})^2 + n_i > n \text{ and } 1^{n_0+1}0^{m_0+1}1^{n_1+1}...0^{m_{i-1}+1}1^{n_i+1}0 \sqsubseteq p_{h(n)}.$$

Thus $h(n)$ finds a path $p_{h(n)}$ which on depth $\gamma(n) + 1$ "tries to prove" that the value $\sigma(n) > n$ is nonrandom. Now let us define a function $f :\subseteq \mathbb{N} \to \mathbb{N}$ by

$f\langle n,0\rangle := \sigma(n)$ and $f\langle n,j+1\rangle = k$, if and only if there exist $n_0,...,n_{\gamma(n)+j}$, $m_0,...,m_{\gamma(n)+j} \in \mathbb{N}$ such that

$$1^{n_0+1}0^{m_0+1}...1^{n_{\gamma(n)}+1}0^{m_{\gamma(n)}+1}...1^{n_{\gamma(n)+j}+1}0^{m_{\gamma(n)+j}+1}1^{k+1}0 \sqsubseteq p_{h(n)}$$

for all $n,j,k \in \mathbb{N}$. Then f is computable too. We mention that $f\langle n,j\rangle$ is undefined for all $n \in \mathbb{N}$ such that $\sigma(n) \in \mathbb{N} \setminus L$ is random and $j > 0$ and it is defined for all $n \in \mathbb{N}$ such that $\sigma(n) \in L$ is nonrandom and a certain initial segment $j = 0,...,\iota$. Since the set $\mathbb{N} \setminus L$ is immune, it follows that $\sigma(n) \in L$ for almost all $n \in \mathbb{N}$. Altogether, the function $g :\subseteq \mathbb{N} \to \mathbb{N}$, defined according to Lemma 9, is computable and there exists some sufficiently large $n \in \mathbb{N}$ such that $\sigma(n) \in L$ and such that the sequence $g\langle n,0\rangle, g\langle n,1\rangle, g\langle n,2\rangle...$ does not contain a random number. For this fixed n there exists some $\iota > 0$ and $n_0,...,n_{\gamma(n)}, m_0,...,m_{\gamma(n)-1}$ such that

$$p_{h(n)} = 1^{n_0+1}0^{m_0+1}...1^{n_{\gamma(n)}+1}0^{tg\langle n,0\rangle+1}1^{f\langle n,1\rangle+1}0^{tg\langle n,1\rangle+1}... \\ 1^{f\langle n,\iota-1\rangle+1}0^{tg\langle n,\iota-1\rangle+1}1^{f\langle n,\iota\rangle+1}0^{\omega} \in T$$

and thus $g\langle n,\iota\rangle \in \mathbb{N} \setminus L$ is random by definition of T. But this is a contradiction to the choice of n! □

If we generalize the notion of constructively immune sets [6] to subsets of computable metric spaces appropriately, then Theorem 11 together with Proposition 7 imply that T is even a constructively immune subset of $\mathcal{Z}$.

4 The rational case

The aim of this section is to transfer the tree T and its properties from Cantor space to Euclidean space. The main idea is to use the transformation which is given in the following lemma.

Lemma 12. *The function $\Gamma : \{0,1\}^\omega \to \mathbb{R}$, $p \mapsto \sum_{i=0}^{\infty} 2^{-2^i} p(i)$ has the following properties:*

1. *Γ is a computable embedding, i.e. Γ is injective and Γ as well as its partial inverse Γ^{-1} are computable.*
2. *Γ preserves "the dense subset", i.e. $\Gamma(\mathcal{Z}) \subseteq \mathbb{Q}$ and $\Gamma(\{0,1\}^\omega \setminus \mathcal{Z}) \subseteq \mathbb{R} \setminus \mathbb{Q}$.*

The stated properties can easily be verified. Now we formulate a proposition which shows that Γ also preserves several effectivity properties of sets.

Proposition 13. *Let $A \subseteq \{0,1\}^\omega$ be a subset. Then the following holds:*

1. *A immune $\iff$ $\Gamma(A)$ immune.*
2. *A effectively separable in $\{0,1\}^\omega$ $\iff$ $\Gamma(A)$ effectively separable in $\mathbb{R}$.*
3. *A r.e. closed in $\{0,1\}^\omega \implies \Gamma(A)$ r.e. closed in $\mathbb{R}$.*
4. *A strongly co-r.e. closed in $\{0,1\}^\omega \implies \Gamma(A)$ strongly co-r.e. closed in $\mathbb{R}$.*
5. *A strongly rec. closed in $\{0,1\}^\omega \implies \Gamma(A)$ strongly rec. closed in $\mathbb{R}$.*

Proof. 1. and 2. immediately follow from the fact that Γ is a computable embedding.
3. Let A be an r.e. closed subset of $\{0,1\}^\omega$. Then the set

$$P := \{w \in \{0,1\}^* : w\{0,1\}^\omega \cap A \neq \emptyset\}$$

is r.e. For all $a, b \in \mathbb{Q}$ we denote by (a, b) the corresponding open interval and we obtain

$$\begin{aligned}(a,b) \cap \Gamma(A) \neq \emptyset &\iff \Gamma^{-1}(a,b) \cap A \neq \emptyset \\ &\iff (\exists w \in P)\ w\{0,1\}^\omega \subseteq \Gamma^{-1}(a,b)\end{aligned}$$

Using a theorem on effective continuity (e.g. Theorem 6.2 in [1]) we can conclude that $\Gamma(A)$ is r.e. closed in $\mathbb{R}$.
4. Let A be a strongly co-r.e. closed subset of $\{0,1\}^\omega$. Then $\{0,1\}^* \setminus P$, with P defined as in 3., is r.e. For all $a, b \in \mathbb{Q}$ we denote by $[a, b]$ the corresponding closed interval and we obtain

$$\begin{aligned}[a,b] \cap \Gamma(A) = \emptyset &\iff \Gamma^{-1}[a,b] \cap A = \emptyset \\ &\iff (\exists \text{ open } U \subseteq \{0,1\}^\omega)\ \Gamma^{-1}[a,b] \subseteq U \text{ and } U \cap A = \emptyset \\ &\iff (\exists w_1, ..., w_n \in \{0,1\}^* \setminus P)\ \Gamma^{-1}[a,b] \subseteq \bigcup_{i=1}^{n} w_i\{0,1\}^\omega.\end{aligned}$$

Here, the last equivalence holds since $\Gamma^{-1}[a, b]$ is compact (as a closed subset of the compact Cantor space $\{0,1\}^\omega$). Using a theorem on effective continuity (Theorem 6.2 in [1]) and the effective Heine-Borel Theorem (Theorem 4.10 (2) in [3]), we can conclude that $\Gamma(A)$ is strongly co-r.e. closed in $\mathbb{R}$.
5. is a direct consequence of 3. and 4. □

Now we formulate a lemma which allows to transfer effectivity properties to subspaces and back.

Lemma 14. *Let $A \subseteq \mathbb{Q}$ be a subset which is closed in $\mathbb{Q}$ and let $\overline{A}$ denote the closure of A in $\mathbb{R}$. Then*

1. *$\overline{A}$ is r.e. closed in $\mathbb{R}$ $\iff$ A is r.e. closed in $\mathbb{Q}$.*
2. *$\overline{A}$ is strongly co-r.e. closed in $\mathbb{R}$ $\iff$ A is strongly co-r.e. closed in $\mathbb{Q}$.*
3. *$\overline{A}$ is strongly recursive closed in $\mathbb{R}$ $\iff$ A is strongly recursive closed in $\mathbb{Q}$.*
4. *A is effectively separable in $\mathbb{R}$ $\iff$ A is effectively separable in $\mathbb{Q}$.*

Proof. 1. For all $a, b \in \mathbb{Q}$ we obtain $(a, b) \cap A \neq \emptyset \iff (a, b) \cap \overline{A} \neq \emptyset$, since (a, b) is open.
2. For all $a, b \in \mathbb{Q}$ we obtain $[a, b] \cap A = \emptyset \iff [a, b] \cap \overline{A} = \emptyset$. Here, on the one hand, "$\Longleftarrow$" follows since $A \subseteq \overline{A}$. And, on the other hand, "$\Longrightarrow$" follows since $[a, b] \cap A = \emptyset$ implies $[a, b] \cap \overline{A} \subseteq \partial[a, b] = \{a, b\} \subseteq \mathbb{Q}$ and thus $[a, b] \cap \overline{A} = \emptyset$ since A is closed in $\mathbb{Q}$.
3. Is a direct consequence of 1. and 2.
4. A sequence $f : \mathbb{N} \to \mathbb{Q}$ is dense in A and computable with respect to $\mathbb{Q}$, if and only if it is dense in A and computable with respect to $\mathbb{R}$ (considered as a sequence $f : \mathbb{N} \to \mathbb{R}$). □

It should be mentioned that our results show that direction "$\Longrightarrow$" of 4. does not hold if A is replaced by its closure $\overline{A}$ on the left-hand side. Finally, we are prepared to prove our main result on $\Gamma(T)$ as a subset of $\mathbb{Q}$.

Theorem 15. *$\Gamma(T)$ is a strongly recursive closed subset of $\mathbb{Q}$, which is immune and thus not effectively separable.*

Proof. On the one hand, $\overline{T}$ is strongly recursive closed in $\{0,1\}^\omega$ by Corollary 8 and thus $\overline{\Gamma(T)} = \Gamma(\overline{T})$ is strongly recursive closed in $\mathbb{R}$ by Proposition 13.5. Hence $\Gamma(T)$ is strongly recursive closed in $\mathbb{Q}$ by Lemma 14.3. since $\Gamma(T)$ is closed in $\mathbb{Q}$. On the other hand, T is immune by Theorem 11 and thus $\Gamma(T)$ is immune by Proposition 13.1. $\square$

One could also directly use Theorem 10, Proposition 13.2. and Lemma 14.4. to conclude that $\Gamma(T)$ is not effectively separable in $\mathbb{Q}$ (without using Theorem 11). The reader should notice that immunity in Theorem 15 means that there is no infinite computable sequence of reals contained in $\Gamma(T)$. This implies the weaker statement that there is no infinite sequence of rationals included in $\Gamma(T)$, which is computable in the classical discrete sense.

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