PSPACE-completeness of the reachability relation of robust dynamical systems

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Dynamical systems are widely used in many areas of fundamental and applied science. Their interactions with computations over the reals have led to numerous works, with various motivations. One of the primary challenges in dynamical systems is the problem of reachability: can a point y be reached from x? This decision problem is well-known to be computably enumerable but not co-computably enumerable, so it is not decidable in the general case. However, with a proper notion of stability, i.e. *robustness* as defined in [1], the reachability relation becomes decidable. Intuitively, it can be understood as insensitivity to an arbitrarily small perturbation. Instead of sending one point to another point, the perturbed dynamic function sends one point to an open set. A system is robust if and only if each of those open sets intersects the non-perturbed solution point.

The authors of [3] proved we can go a bit further, and deal with questions of complexity. The idea of their paper is that with the proper quantification over the allowed perturbation, namely the radius of the open sets is $2^{-p(n)}$ with p a polynomial and $n \in \mathbb{N}$ related to some notion of size, we can prove that the reachability relation is in **PSPACE**.

Also, we are interested in complexity classes working over the real numbers and properly defining our model of computation. Many different models exist for real numbers. For discrete-time models of computations over the reals, we consider here computable analysis, based on the Turing machine model in [7] and [9]. Using this framework, the authors of [2] prove it is possible to have algebraic characterisations of **PTIME** and **PSPACE**, relying on discrete ODEs.

For continuous time models, one of the first machines is the first-ever built computer, such as the Differential Analysers [8]. It was mathematically formalised by the General Purpose Analog Computer (GPAC) model of Claude Shannon [6]. Going back to Turing machines, the authors of [4] prove it is possible to have algebraic characterisations of **PSPACE**, relying on continuous ODEs. In [5], the authors even have a **PSPACE**-completeness result, by restricting themselves to compact domains.

Contributions. There exist **PSPACE**-completeness results, for **PSPACE** over the reals, but in the case where the domain of the functions are compact sets, as in [5]. We prove a **PSPACE**-completeness property for (not necessarily) bounded real domain.

We prove that **PSPACE**-completeness in the framework of [2], where the domain of our dynamical systems is included in $\mathbb{Q}$ and is robust, with a proper notion of robustness. We also show here that, with a proper notion of robustness

for a particular kind of Turing machines, namely Type-2 machines, and second-order reductions over real functions, as defined in [5], we have the $\mathbf{PSPACE}^2$-$\leq_m^2$-completeness of the reachability relation. We are no longer in the previous framework, allowing us to deal with somewhat more general dynamic functions.

References

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