CCA 2025 Kyoto

Abstracts

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CCA 2025 Schedule (Time zone: Asia/Tokyo)

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1 Invited Talks

Analysing Turing degree over intuititionistic logic

Takako Nemoto

Graduate School of Information Sciences, Tohoku University

It is known that a basic part of recursion theory (cf. [1]), such as recursion theory, smn-theorem, recursion theorem and normal form theorem can be formalized in **HA** [2, Ch.3.7]. But how advance we can develop recursion theory over intuitionistic logic?

In this talk, we analyze, over an intuitionistic system, what non-constructive principles and induction principles are enough to show some properties of the Turing degree, including the existence of simple sets, simple and low sets, two incomparable degrees, and so on.

References

[1] R. Soare, *Recursively Enumerable Sets and Degrees*, Springer-Verlag, 1987.

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Comparing relative information content

Selwyn Ng
Nanyang Technological University

Abstract

The notion of a Turing reduction is one of the central notions in computability theory. The robustness of this notion means that it provides the standard tool for analysing the algorithmic content of different objects. However there are different reducibilities which may be more suitable in a different setting. We will discuss several - perhaps less well-known - reducibilites and some survey recent results about them.

TLPP and its Surroundings

Yudai Suzuki
National Institute of Technology, Oyama College

Abstract

Towsner's transfinite leftmost path principle TLPP plays a central role to study the complexity of theorems whose complexity lies between ATR_0 and Π^1_1-CA_0 in the context of reverse mathematics. In this talk, I will consider some theorems related to TLPP from the point of view of reverse mathematics and computability-theoretic reductions.

▸ PATRICK UFTRING, *Computable transformations of Turing degrees.*
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Following a conjecture by Vasco Brattka, we present a new result that fully characterizes all partial, multi-valued functions on the Turing degrees that have computable realizers. To be precise, for any n-ary such function $f : \mathcal{D}^n \rightrightarrows \mathcal{D}$, there is a set $N \subseteq n$ such that $\bigvee_{i \in N} \mathbf{a}_i \in f(\mathbf{a}_0, \dots, \mathbf{a}_{n-1})$ holds for any tuple of Turing degrees $(\mathbf{a}_0, \dots, \mathbf{a}_{n-1}) \in \mathsf{dom}(f)$. This yields short proofs of established results such as [2, Proposition 11.5] ("The inverse of the Turing jump is not computable.") but also answers open questions such as "Are two applications of PA reducible to a single instance?" (private communication with Vasco Brattka, cf. [1, Proposition 38]). We also present an infinitary variant, which can e.g. be applied to Weihrauch reductions involving infinite suborders of Turing degrees and answers [1, Conjecture 37].

[1] Vasco Brattka, *Loops, Inverse Limits and Non-Determinism*, arXiv:2501.17734, pp. 1–18.

[2] Vasco Brattka, Matthew Hendtlass, Alexander P. Kreuzer, *On the Uniform Computational Content of Computability Theory*, ***Theory of Computing Systems***, vol. 61 (2017), no. 4, pp. 1376–1426.

2 Contributed Talks

▸ THIBAUT KOUPTCHINSKY, *The reverse mathemtaics of analytic measurability.*.
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This talk is about a work in progress with Juan Aguilera (TU Wien) and Keita Yokohama (Tohoku University) on the foundations of mathematics, studying measurability of Σ^1_1 set, with the method of random forcing. Reverse Mathematics is the branch of Mathematical Logic which studies the question: "given a mathematical theorem ϕ, which axioms are necessary to prove ϕ?", therefore providing a comparison of the theorems of mathematics according to some measure of their provability strength (see [2]).

We study a theorem of Lusin [1], which stated that all analytical set of reals are Lebesgue-measurable. We work in the framework of second-order arithmetic. Yu [4] showed that assuming transfinite recursion, ATR_0 is sufficient (and necessary) to prove that $\lambda(A)$ exists for all Borel sets A, coded as wellfounded trees on natural numbers. Simpson [2] asked whether ATR_0 "suffices to prove measurability and regularity of analytic sets in some appropriate sense." The purpose of our work is to answer this question.

The following is our main theorem:

THEOREM 1. *The following are equivalent over* ATR_0.

1. *All analytic sets* $A \subset [0,1]$ *are Lebesgue-regular; and*
2. $\mathbf{\Sigma}^1_1$*-Induction for* $\mathbb{N}$.

In addition to answering Simpson's question, Theorem 1 presents a reversal of a theorem to $\mathbf{\Sigma}^1_1$-Induction. We do not know of any other theorem from core mathematics equivalent to this theory.

Theorem 1 deals with Lebesgue-regularity, i.e., the equality of the outer and inner measures. If one additionally demands that this value exist as a number, the strength of Lusin's theorem increases:

THEOREM 2. *The following are equivalent over* ATR_0.

1. *All analytic sets* $A \subset [0,1]$ *are Lebesgue-measurable; and*
2. $\mathbf{\Pi}^1_1$*-Comprehension.*

The main idea is to draw inspiration from Solovay's [3] construction of a model of Zermelo-Fraenkel set theory where every set is Lebesgue measurable. In our case the forcing argument is carried out over a non-standard model of a weak set theory (obtained through the familiar method of *pseudohierarchies*). The main subtle point is the use of $\mathbf{\Sigma}^1_1$-Induction to guarantee that the forcing provides enough information about a given analytic set A to conclude that $\lambda^*(A) = \lambda_*(A)$.

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[2] S. G. SIMPSON, ***Subsystems of second order arithmetic*** (Second edition), Perspectives in Logic, Association for Symbolic Logic, 2009.

[3] R. SOLOVAY, *A model of set theory in which every set of reals is Lebesgue measurable* ***Ann. Math.*** 92(1):1–56, 1970.

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PSPACE-completeness of the reachability relation of robust dynamical systems

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Dynamical systems are widely used in many areas of fundamental and applied science. Their interactions with computations over the reals have led to numerous works, with various motivations. One of the primary challenges in dynamical systems is the problem of reachability: can a point y be reached from x? This decision problem is well-known to be computably enumerable but not co-computably enumerable, so it is not decidable in the general case. However, with a proper notion of stability, i.e. *robustness* as defined in [1], the reachability relation becomes decidable. Intuitively, it can be understood as insensitivity to an arbitrarily small perturbation. Instead of sending one point to another point, the perturbed dynamic function sends one point to an open set. A system is robust if and only if each of those open sets intersects the non-perturbed solution point.

The authors of [3] proved we can go a bit further, and deal with questions of complexity. The idea of their paper is that with the proper quantification over the allowed perturbation, namely the radius of the open sets is $2^{-p(n)}$ with p a polynomial and $n \in \mathbb{N}$ related to some notion of size, we can prove that the reachability relation is in **PSPACE**.

Also, we are interested in complexity classes working over the real numbers and properly defining our model of computation. Many different models exist for real numbers. For discrete-time models of computations over the reals, we consider here computable analysis, based on the Turing machine model in [7] and [9]. Using this framework, the authors of [2] prove it is possible to have algebraic characterisations of **PTIME** and **PSPACE**, relying on discrete ODEs.

For continuous time models, one of the first machines is the first-ever built computer, such as the Differential Analysers [8]. It was mathematically formalised by the General Purpose Analog Computer (GPAC) model of Claude Shannon [6]. Going back to Turing machines, the authors of [4] prove it is possible to have algebraic characterisations of **PSPACE**, relying on continuous ODEs. In [5], the authors even have a **PSPACE**-completeness result, by restricting themselves to compact domains.

Contributions. There exist **PSPACE**-completeness results, for **PSPACE** over the reals, but in the case where the domain of the functions are compact sets, as in [5]. We prove a **PSPACE**-completeness property for (not necessarily) bounded real domain.

We prove that **PSPACE**-completeness in the framework of [2], where the domain of our dynamical systems is included in $\mathbb{Q}$ and is robust, with a proper notion of robustness. We also show here that, with a proper notion of robustness

for a particular kind of Turing machines, namely Type-2 machines, and second-order reductions over real functions, as defined in [5], we have the $\mathbf{PSPACE}^2$-$\leq_m^2$-completeness of the reachability relation. We are no longer in the previous framework, allowing us to deal with somewhat more general dynamic functions.

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Non-deterministic analogue computations with ODEs: Towards a characterisation of NP

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In 1941, Claude Shannon introduced in [12] a model of analogue computation, the GPAC (general purpose analogue computer). This was proposed as a mathematical model of (mechanical) analogue computers of that time, namely differential analysers. In this model, we can compose basic blocs (constants, additions, multiplications, integrators) into circuits (feedbacks are allowed) in order to compute functions. A GPAC can also be thought of as computing solutions of polynomial initial value problems (PIVPs), a particular class of ordinary differential equations. A characterisation in that direction was obtained in [5]. In terms of computability, it was first thought to be a less powerful model than the Turing machine model, but it is now regarded as equivalent, as long as we use a coherent definition of computation for this model, following [3]. In practice, from strong closure properties, it can be considered as a model of any (old or modern) analogue computing device.

The question of complexity was addressed only recently by [11,4], who proved that the time complexity of the computation corresponded to the length of the curve of the solution of the ODE associated with the GPAC. More precisely, a real function is computable if and only if it is computable by a GPAC of polynomial length [11,4]. Very recently, it was also proved that space corresponds to precision [1,2]. All these results, providing analogue characterisations of P, and PSPACE at the end, were obtained by proving both that Turing machines computations could be simulated by various classes of ordinary differential equations in one direction, and arguments from computable analysis in the other direction. In particular, the statement about the time complexity is highly based on the fact that there is a way to solve ordinary differential equations in a time that is polynomial in the length of the solution.

In the more traditional computable analysis approach, the complexity of solving ODEs has been investigated by many works [8,9,10,6]. Concerning analytic functions, the work by [13,7] showed that we could compute solutions of slightly more general problems than PIVPs, namely IVPs with analytic right-hand sides. The approach is based on parametrised complexity, where some parameters are assumed to be encoded in unary, while others have their usual representation in binary [13,7]. There, the length of the curve was not used, but instead specific parametric representations that would give the information needed to make evaluation and ODE-solving polynomial-time computable. These representations each consist of a way to represent the analytic function itself (by an approxima-

tion of the function or by the coefficients of the associated series) and of some bounds on natural quantities, which can be interpreted as radii of convergence, that make the operations that we want to perform polynomial-time computable. As noticed by both authors, the (parametrised) algorithms that they proposed have similarities, even if both admit that this requires more investigation.

One first outcome of this work is to explain the relations between both. More generally, the goal of the present work is motivated by extending previous statements about analogue computations, in order to deal with non-determinism: can we provide a characterisation of the NP class, using ordinary differential equations in the spirit of the above analogue characterisations of P and PSPACE? Concretely, an idea is that by adding a discrete parameter in a discrete-time system, one can simulate non-determinism: a non-deterministic computation succeeds if the computation succeeds for some value of the parameter, which means performing an existential quantification on the parameter. This provides one of the directions. Conversely, the idea is that by using approximation theory by analytic functions and even polynomials, one can add such a parameter in a continuous-time system, and under some conditions, the good behaviour and complexity are preserved. If this idea is simple, the deep key point is to identify the class of allowed functions or dynamics that may be authorised in the quantification, in order to remain in the class NP.

The proposed work identifies a suitable class of such dynamics. We unify the parameters from [13] (the radius of convergence over the reals and a bound for the first representation, the size of a subdomain over the complex and another bound for the second one) and the length from [11] to get a better understanding of the quantity that we have to bound to guarantee polynomial-time computability. As another side effect, we also identify the role played by absolute continuity in involved reasoning and representations.

This work corresponds to the master's thesis of the second author (in alphabetical order). His report will be available on: `https://www.lix.polytechnique.fr/~bournez/CCA2025/` as soon as possible. A preliminary version (in French) is already available at this url.

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COMPUTABLE BASES

VASCO BRATTKA AND EMMANUEL RAUZY

Abstract. In computable analysis typically topological spaces with countable bases are considered. The Theorem of Kreitz-Weihrauch implies that the subbase representation of a second-countable T_0 space is admissible with respect to the topology that the subbase generates. We consider generalizations of this setting to bases that are representable, but not necessarily countable. We introduce the notions of a computable presubbase and a computable prebase. We prove a generalization of the Theorem of Kreitz-Weihrauch for the presubbase representation that shows that any such representation is admissible with respect to the topology generated by compact intersections of the presubbase elements. For computable prebases we obtain representations that are admissible with respect to the sequentialization of the topology that they generate. These concepts provide a natural way to investigate many topological spaces that are studied in computable analysis.

1. Summary

We start with defining the concept of a presubbase.

Definition 1.1 (Presubbase)**.** Let X be a set. We call a family $(B_y)_{y\in Y}$ a *presubbase* for X, if Y is a represented space and its *transpose*

$$B^{\mathsf{T}} : X \to \mathcal{O}(Y), x \mapsto \{y \in Y : x \in B_y\}$$

is well-defined and injective.

Injectivity of B^{T} implies that $(B_y)_{y\in Y}$ is a subbase of some T_0 topology on X. We note that every countable subbase $B : \mathbb{N} \to \mathcal{O}(X)$ of a T_0 topology is a particular instance of a presubbase, as $B^{\mathsf{T}} : X \to \mathcal{O}(\mathbb{N})$ is always well-defined. Hence, the following definition generalizes the concept of a subbase representation as it is known in computable analysis [Wei00].

Definition 1.2 (Presubbase representation)**.** Let $(B_y)_{y\in Y}$ be a presubbase of a set X. We define the *presubbase representation* $\delta^B :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ by

$$\delta^B(p) = x :\iff \delta_{\mathcal{O}(Y)}(p) = \{y \in Y : x \in B_y\}$$

for all $p \in \mathbb{N}^{\mathbb{N}}$ and $x \in X$.

The reason that we speak about a *pre*subbase representation in this general situation and not about a subbase representation is that δ^B is not necessarily admissible with respect to the topology generated by $(B_y)_{y\in Y}$. However, it is admissible with respect to a closely related topology generated by compact intersections of the sets B_y, as shown in the next theorem. We call a represented space X a *computable Kolmogorov space* if its neighborhood map $\mathcal{U} : X \to \mathcal{O}\mathcal{O}(X), x \mapsto \{U \in \mathcal{O}(X) : x \in U\}$ is a computable embedding. This is the natural effectivization of the T_0 property (that is sometimes called *computable admissibility*) [Sch02, Sch21].

Theorem 1.3 (Presubbase theorem)**.** *Let $(B_y)_{y\in Y}$ be a presubbase of a set X. Then (X, δ^B) is a computable Kolmogorov space and δ^B is admissible with respect to the topology τ on X with the base sets X and $\bigcap_{y\in K} B_y$ for every compact $K \subseteq Y$.*

We note that this result generalizes the Theorem of Kreitz-Weihrauch [KW85] as for countable subbases $B : \mathbb{N} \to \mathcal{O}(X)$ the compact subsets $K \subseteq \mathbb{N}$ are exactly the finite subsets and hence the topology generated by X and $\bigcap_{n\in K} B_n$ for compact $K \subseteq \mathbb{N}$ is exactly the same topology as the topology generated by the subbase B itself. We can now define the notion of a computable presubbase.

Definition 1.4 (Computable presubbase)**.** Let X and Y be represented spaces. Then $B : Y \to \mathcal{O}(X)$ is called a *computable presubbase* of X if the transpose

$$B^{\mathsf{T}} : X \to \mathcal{O}(Y), x \mapsto \{y \in Y : x \in B_y\}$$

is well-defined and a computable embedding.

Obviously, a presubbase B of X is a computable presubbase of X if and only if the representation of X is computably equivalent to δ^B. In analogy to the countable case [BR25] we can now define the concept of a computable base.

Definition 1.5 (Computable prebase)**.** Let X a represented space with a computable presubbase $B : Y \to \mathcal{O}(X)$. Then B is called a *computable prebase* of X, if there is a computable $R : \mathcal{K}_-(Y) \rightrightarrows \mathcal{A}_+(Y)$ such that $\bigcap_{y\in K} B_y = \bigcup_{y\in A} B_y$ for every $K \in \mathcal{K}_-(Y)$ and $A \in R(K)$ and $X = \bigcup_{y\in Y} B_y$. We call B a *computable base* of X if B is actually a base of X

By Theorem 1.3 computable prebases characterize the topology of their spaces up to sequentialization, which one can see using results of Schröder [Sch02]. In fact, every computable Kolmogorov space has a computable base, namely the identity. Altogether, we obtain the following characterization of computable Kolmogorov spaces in terms of their bases.

Theorem 1.6 (Computable Kolmogorov spaces and bases)**.** *Let X be a represented space X. Then the following are pairwise equivalent:*

(1) X is a computable Kolmogorov space,
(2) X has a computable presubbase,
(3) X has a computable prebase,
(4) X has a computable base,
(5) $\mathrm{id} : \mathcal{O}(X) \to \mathcal{O}(X)$ is a computable base of X.

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Speedability of computably approximable reals and their approximations

Nan Fang

Abstract

Given a computable approximation $\{a_s\}_{s\in\omega}$ to a real α, we say the approximation is *speedable* if there exists a computable function f such that the modified approximation $\{a_{f(s)}\}_{s\in\omega}$ converges faster than $\{a_s\}_{s\in\omega}$. This leads to various notions of *speedability* for computably approximable reals, depending on the speedability of their computable approximations, and the computational complexity of the reals as well.

Left/right-c.e. reals are those with nondecreasing/nonincreasing computable approximations, respectively; d.c.e. reals are differences of two left-c.e. reals; and computably approximable reals are those that admit a computable approximation. In this talk, we examine speedability notions in the context of these different classes of reals. Previous work by Merkle and Titov established the equivalence of several speedability notions for left-c.e. reals. For right-c.e. reals, the situation is symmetrical. We extend these results to d.c.e. reals, showing that the various notions of speedability coincide in this broader setting as well. Moreover, we prove that for d.c.e. reals, being speedable is equivalent to not being Martin-Löf random. Finally, for the general class of computably approximable reals, we show that every such real admits at least one speedable computable approximation.

This is joint work with George Barmpalias, Wolfgang Merkle, and Ivan Titov.

DedekindCutArithmetic.jl: A Julia implementation of exact real arithmetic based on Dedekind cuts

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In CCA 2008, Andrej Bauer introduced Marshall (Bauer 2008), a programming language written in OCaml for exact real arithmetic based on Abstract Stone Duality (Bauer and Taylor 2009). At the end of the presentation, a challenge was thrown to the audience, *Can this be implemented as library in an existing programming language?*. Today, we show the answer is yes.

This talk introduces DedekindCutArithmetic.jl, a library that allows for computations with exact reals. Leveraging Julia's lisp-style syntactic macros, `DedekindCutArithmetic.jl` offers an embedded domain specific language (eDSL), which allows to write cuts and expressions with quantifiers in a notation resembling the traditional mathematical one.

Since it is an eDSL, it canc ompose with to other libraries in the Julia ecosystem. For example, similarly to Marshall, the library relies on interval Newton method to speed up refinement of cuts and quantifiers. However, the derivative is computed using `ForwardDiff.jl` (Revels, Lubin, and Papamarkou 2016), a library for high-performance forward-mode automatic differentiation. As an additional advantage, being the language embedded in Julia, `DedekindCutArithmetic.jl` automatically inherits all the features of the language, such as support for higher-order functions and recursion, without the need to be re-implemented.

During this talk, I will introduce the features of `DedekindCutArithmetic.jl`, focusing on its architecture, design choices and trade-offs. I will also demostrate in a few examples how the library can be composed with other libraries in the Julia ecosystem, to be applied to computational geometry and scientific computing domains. Finally, I will also give an overview of the roadmap for the library, describing current limitations, next steps and open-questions.

Since few lines of code generally say more than 1000 words, the following copy-pastable working example demonstrates the syntax and some core functionalities of the library.

```julia
using DedekindCutArithmetic

# Textbook example of dedekind cuts, define square-root
my_sqrt(a) = @cut x ∈ ℝ, (x < 0) ∨ (x * x < a), (x > 0) ∧ (x * x > a)

sqrt2 = my_sqrt(2);

# evaluate to 80 bits precision, this gives an interval with width <2^-80 containing √2
refine!(sqrt2; precision=80)
# [1.4142135623730949, 1.4142135623730951]

# Define maximum of a function f: [0, 1] → ℝ as a Dedekind cut
my_max(f::Function) = @cut a ∈ ℝ, ∃(x ∈ [0, 1] : f(x) > a), ∀(x ∈ [0, 1] : f(x) < a)

f = x -> x * (1 - x)

fmax = my_max(f);

refine!(fmax) # evaluate to 53 bits of precision by default
# [0.24999999999999992, 0.25000000000000006]
```

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Subrecursive degrees of difference representations of irrational numbers

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Abstract. We explore the degree structure of representations of irrational numbers, induced by subrecursive reducibility. The representation R_1 is subrecursive in the representation R_2, denoted $R_1 \preceq_S R_2$, if there exists a Turing operator, which transforms any R_2-representation of an irrational α into some R_1-representation of the same α. The operator is not allowed to use unbounded minimization and it must work uniformly in α.

For example, we have $\mathcal{C} \prec_S \mathcal{E}_b \prec_S \mathcal{D} \prec_S [\,]$, where $\mathcal{C}$ is the representation by Cauchy sequences with fixed convergence rate, $\mathcal{E}_b$ is the representation by base-b expansions, $\mathcal{D}$ is the representation by Dedekind cuts and $[\,]$ is the representation by continued fractions.

This degree structure was formally introduced in [1] quite recently, but the study of concrete representations in this context can be traced back to the foundational papers [7,8] and since then it has been constantly evolving, see for example [2,3,5,6].

In this talk we will present a novel idea to define new representations based on difference from rational numbers. Somewhat surprisingly, this idea produces numerous new subrecursive degrees.

Let R be a representation of the real numbers, such that any $\alpha \in [0,1]$ has a unique representation R^α. We assume $R^\alpha : A \to B$, where A, B are fixed primitive recursively encoded countable sets. For any $\alpha \in (0,1) \setminus \mathbb{Q}$ we define:

$$\mathcal{D}\mathrm{iff}_R^\alpha(q) \;=\; \mu a \in A[R^\alpha(a) \neq R^q(a)],$$

where the minimum is regarded with respect to the coding of A.

We will need some assumptions, so that $\mathcal{D}\mathrm{iff}_R$ is a representation in the sense of [1], that is $\mathcal{D}\mathrm{iff}_R^\alpha$ is uniformly computably equivalent to the Dedekind cut $\mathcal{D}^\alpha$:

1. The restriction of R to rational numbers, $R : \mathbb{Q} \times A \to B$, must be computable.
2. There exists a binary computable function P, such that whenever $\overline{a}$ is a finite list of elements of A and $\overline{b}$ is a finite list of elements of B with the same length, $P(\overline{a}, \overline{b})$ produces a rational number q, which satisfies $R^q(a_i) = b_i$ for all i.

The first assumption implies that we can compute $\mathcal{D}\mathrm{iff}_R^\alpha$ using oracle R^α (in general, unbounded search is needed for that).

The second assumption allows computation of $R^\alpha(a)$ by the following algorithm:

1. Compute all elements $a_0, \ldots, a_k = a$ of A with code at most the code of a. Assume that we have inductively computed $R^\alpha(a_i)$ for $i < k$.

2. For each $b_k \in B$:

   Compute $q = P(\overline{a}, \overline{b})$, where $b_i = R^\alpha(a_i)$ for $i < k$.

   If $\mathcal{D}\mathrm{iff}_R^\alpha(q) \neq a$, then return output $R^\alpha(a) = b_k$.

This is a computable reduction from $\mathcal{D}\mathrm{iff}_R^\alpha$ to R^α, therefore $\mathcal{D}\mathrm{iff}_R^\alpha$ is computably equivalent to R^α. So whenever R is a representation in the sense of [1], $\mathcal{D}\mathrm{iff}_R$ is also a representation.

Note that in case B is finite and P is primitive recursive no unbounded search is needed in the above algorithm. But if B is infinite, $\mathcal{D}\mathrm{iff}_R$ and R might even turn out to be subrecursively incomparable.

As an example, let us consider the representation $\mathcal{D}\mathrm{iff}_b$, which corresponds to the base-b expansion $\mathcal{E}_b$. The base-b expansions of numbers of the form $\frac{m}{b^n}$ are assumed to end in zeros.

The base-b sum approximation from below $\hat{A}_b^\alpha$ of $\alpha \in [0,1]$ is defined as $\hat{A}_b^\alpha(n) = \mathtt{D}_n b^{-k_n}$, where $\mathtt{D}_n \neq 0$ and $\alpha = \sum_{n=0}^{\infty} \mathtt{D}_n \cdot b^{-k_n}$. The base-$b$ sum approximation from above $\check{A}_b^\alpha$ is defined symmetrically, so that $\check{A}_b^\alpha(n) = \hat{A}_b^{1-\alpha}(n)$. The base-$b$ sum approximations of rational numbers with finite base-b expansion are assumed to end in zeros. The general sum approximation from below (from above) of $\alpha \in [0,1]$ is defined as $\hat{G}^\alpha(b,n) = \hat{A}_b^\alpha(n)$ ($\check{G}^\alpha(b,n) = \check{A}_b^\alpha(n)$).

In the paper [4], it is shown that $\hat{G} \prec_S \mathcal{D}\mathrm{iff}_b \prec_S [\,]$.

Our aim here is to present some new results for the representation $\mathcal{D}\mathrm{iff}_{b\uparrow}$ corresponding to base-b sum approximations from below (symmetric results hold for $\mathcal{D}\mathrm{iff}_{b\downarrow}$).

Theorem 1. $\mathcal{E}_b \preccurlyeq_S \mathcal{D}\mathrm{iff}_{b\uparrow}$ and $\mathcal{D} \preccurlyeq_S \mathcal{D}\mathrm{iff}_{b\uparrow}$.

Theorem 2. $\mathcal{D}\mathrm{iff}_{b\uparrow} \not\preccurlyeq_S \hat{G}$ and $\mathcal{D}\mathrm{iff}_{b\uparrow} \not\preccurlyeq_S \check{G}$.

Theorem 3. $\hat{A}_b \not\preccurlyeq_S \mathcal{D}\mathrm{iff}_{b\uparrow}$ and $\check{A}_b \not\preccurlyeq_S \mathcal{D}\mathrm{iff}_{b\uparrow}$.

Keywords: representations of irrational numbers, subrecursive reducibility, difference representations

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Constant-depth approximation of nowhere-differentiable functions

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1 Introduction

In his 1962 ICM lecture [Kol62], A.N. Kolmogorov proposed a program of approximating continuous functions by Boolean circuits and using circuit size to measure the complexity of approximation. In particular, circuits are viewed as functions on signed dyadic rationals. For a class X of real-valued functions on $[-1,1]$ let $K(X,\epsilon)$ denote the smallest n such that every $f \in X$ is ϵ-approximated by a circuit of size n. Kolmogorov cites work of Ofman giving a tight characterization of $K(X,\epsilon)$ when X is a certain class of smooth functions and upper and lower bounds on $K(X,\epsilon)$ when X is a certain class of functions with an analytic continuation. Here we will cite improvements of these characterizations given by Asarin [Asa84]. Let $W_{p+\alpha,c}$, $(p \geq 0,\ 0 < \alpha < 1)$ denote the class of functions $f : [-1,1] \to [-1,1]$ that are p times continuously differentiable and whose p-th derivative satisfies a Hölder condition with exponent α and coefficient c:

$$|f^{(p)}(x) - f^{(p)}(y)| \leq c|x-y|^{\alpha} \quad \text{for any } x, y \in [-1,1]$$

and let $A_{r,c}$ denote the functions $f : [-1,1] \to \mathbb{R}$ that can be continued analytically to the Bernstein ellipse with foci -1 and 1 and sum of semi-axes r with modulus bounded by c on the ellipse. Then

$$K(W_{q,c},\epsilon) = \Theta\left(\left(\frac{1}{\epsilon}\right)^{\frac{1}{q}} \log\frac{1}{\epsilon}\right)$$

$$K(A_{r,c},\epsilon) = \Omega\left(\log^2\frac{1}{\epsilon} / \log\log\frac{1}{\epsilon}\right) \qquad K(A_{r,c},\epsilon) = \tilde{O}\left(\log^2\frac{1}{\epsilon}\right)$$

The upper bounds are obtained constructively while the lower bounds rely on entropy arguments (Asarin [Asa84] also gives a constructive exponential lower bound for 2^{-n}-approximating a 1-Lipschitz function using an incompressibility-based argument.) Note that in the analytic case, the upper bound gives a 2^{-n}-approximation by size $\tilde{O}(n^2)$ circuits. Also note that $K(W_1, 2^{-n}) = \Theta(n2^n)$, where $W_1 = \bigcup_c W_{1,c}$.

An important aspect of the classical theory of polynomial approximation is the existence of so-called "inverse theorems", roughly stating that being "well-approximated" by low-degree polynomials implies smoothness (or even analytic continuation when convergence is rapid enough) [Ber12]. Kolmogorov observes that corresponding results for small circuit size do not hold. In particular, he notes that a variant of van der Waerden's nowhere-differentiable function can be 2^{-n} approximated by circuits of size $O(n^2)$. Kolmogorov's observation is the starting point for the current note. In particular, we ask whether the lack of an inverse theorem also holds in the case of circuit *depth*, recalling that certain classes of analytic functions may be approximated by constant-depth threshold circuits ([RT92, MT99].) Unfortunately, in this setting we also obtain a negative result by showing that Takagi's nowhere-differentiable function can be 2^{-n} approximated in TC^0 and by defining a variant of the function that can be 2^{-n} approximated in AC^0.

2 Takagi's Function

In the sequel, we will work over $[0,1]$, which is more appropriate for the functions we consider. For $x \in \mathbb{R}$, let $\langle\langle x \rangle\rangle$ denote the distance from x to the nearest integer. In 1901, T. Takagi [Tak01] introduced a continuous

nowhere-differentiable function $\tau : [0,1] \to [0,1]$, which may be defined as follows (see [Lag11] for a complete discussion):

$$\tau(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} \langle\!\langle 2^k x \rangle\!\rangle.$$

Suppose $x, \tilde{x} \in [0,1]$ where $\tilde{x} = \sum_{i=1}^{2n+2} b_i 2^{-i}$ and $|x - \tilde{x}| \leq 2^{-(2n+2)}$. We have

$$\begin{aligned}
|\tau(x) - \tau(\tilde{x})| &= \left| \sum_{k=0}^{\infty} \frac{1}{2^k} \langle\!\langle 2^k x \rangle\!\rangle - \sum_{k=0}^{\infty} \frac{1}{2^k} \langle\!\langle 2^k \tilde{x} \rangle\!\rangle \right| \\
&= \left| \sum_{k=0}^{\infty} \frac{1}{2^k} \langle\!\langle 2^k x \rangle\!\rangle - \sum_{k=0}^{2n+1} \frac{1}{2^k} \langle\!\langle 2^k \tilde{x} \rangle\!\rangle \right| \\
&\leq \sum_{k=0}^{2n+1} \frac{1}{2^k} \left| \langle\!\langle 2^k x \rangle\!\rangle - \langle\!\langle 2^k \tilde{x} \rangle\!\rangle \right| + \sum_{k=2n+2}^{\infty} \frac{1}{2^k} \langle\!\langle 2^k x \rangle\!\rangle \\
&\leq \sum_{k=0}^{2n+1} \frac{1}{2^k} \cdot \frac{1}{2^{2n-k+2}} + \sum_{k=2n+2}^{\infty} \frac{1}{2^{k+1}} \\
&= \frac{2n+2}{2^{2n+2}} + \frac{1}{2^{2n+2}} \leq \frac{1}{2^n}
\end{aligned}$$

So $\tau(\tilde{x})$ provides a $2^{-O(n)}$ approximation to $\tau(x)$. We note that $\tau(\tilde{x})$ is obtained by adding $2n+2$ numbers of $2n+2$ bits, where each number is obtained via a constant-depth transformation of $\tilde{x}$ involving only Boolean operations (see the discussion in the following paragraph.) Since such sums may be computed in TC^0 (see, e.g., [Vol99]), we conclude that τ may be $2^{-O(n)}$-approximated in TC^0.

Could it be possible that τ is approximated by an even simpler class of circuits? We will show that τ cannot be 2^{-n}-approximated in AC^0. To start, we recall [Lag11] that for $x \in [0,1]$

$$\langle\!\langle x \rangle\!\rangle = \begin{cases} x & \text{if } 0 \leq x < \frac{1}{2} \quad \text{i.e., } b_1 = 0 \\ 1 - x & \text{if } \frac{1}{2} \leq x \leq 1 \quad \text{i.e., } b_1 = 1. \end{cases}$$

Furthermore, if $x = \sum_{i=1}^{\infty} b_i 2^{-i} = 0.b_1 b_2 b_3 \ldots$ and $i \geq 0$, we have

$$\langle\!\langle 2^i x \rangle\!\rangle = \begin{cases} 0.b_{i+1} b_{i+2} \ldots & \text{if } b_{i+1} = 0 \\ 0.\bar{b}_{i+1} \bar{b}_{i+2} \ldots & \text{if } b_{i+1} = 1 \end{cases}$$

where for $b \in \{0,1\}$, $\bar{b} = 1 - b = b$. Using the fact that $\bar{b} = b \oplus 1$, where $\oplus$ denotes exclusive-or, we can rewrite this as

$$\langle\!\langle 2^i x \rangle\!\rangle = 0.(b_{i+1} \oplus b_{i+1})(b_{i+1} \oplus b_{i+2}) \ldots$$

and so

$$\frac{1}{2^i} \langle\!\langle 2^i x \rangle\!\rangle = 0.\underbrace{00\ldots0}_{i}(b_{i+1} \oplus b_{i+1})(b_{i+1} \oplus b_{i+2}) \ldots$$

Suppose that we have a family $\{C_n\}$ of constant-depth poly-size Boolean circuits, $C_n : \{0,1\}^{\alpha(n)} \to \{0,1\}^{\beta(n)}$ such that for all $x, \tilde{x} \in [0,1]$ where $\tilde{x} = \sum_{i=1}^{\alpha(n)} b_i 2^{-i}$ and $|x - \tilde{x}| \leq 2^{-\alpha(n)}$, we have $|\tau(x) - C_n(\tilde{x})| \leq 2^{-n}$, where $C_n(\tilde{x})$ denotes $C_n(b_1 b_2 \ldots b_{\alpha(n)})$. In particular, $|\tau(\tilde{x}) - C_n(\tilde{x})| \leq 2^{-n}$, which means that $\tau(\tilde{x})$ and $C_n(\tilde{x})$ must agree on their first n bits. Assuming $\alpha(n) \geq n+1$, consider $\tilde{x}$ where $b_{n+1} = 1$ and $b_i = 0$ for $i > n+1$. From the discussion above we have, for $0 \leq k < n$,

$$\langle\!\langle 2^k \tilde{x} \rangle\!\rangle = \begin{cases} 0.b_{k+1} \ldots b_{n-1} b_n 100 \ldots & \text{if } b_{k+1} = 0 \\ 0.\bar{b}_{k+1} \ldots \bar{b}_{n-1} \bar{b}_n 011 \cdots = 0.\bar{b}_{k+1} \ldots \bar{b}_{n-1} \bar{b}_n 100 \ldots & \text{if } b_{k+1} = 1 \end{cases}$$

From this we see that for $0 \le k < n$, the nth bit of $\frac{1}{2^k}\langle\!\langle 2^k \tilde{x} \rangle\!\rangle$ is $b_{k+1} \oplus b_n$, while the $(n+1)$st bit is always 1, so that

$$\begin{array}{rccccc} \tau(\tilde{x}) = 0.(b_1 \oplus b_1) & (b_1 \oplus b_2) & \dots & (b_1 \oplus b_n) & 1 \\ + \ 0. \quad 0 & (b_2 \oplus b_2) & \dots & (b_2 \oplus b_n) & 1 \\ + \ 0. \quad 0 & 0 & \dots & (b_3 \oplus b_n) & 1 \\ & \dots & & & \\ + \ 0. \quad 0 & 0 & \dots & (b_n \oplus b_n) & 1 \\ + \ 0. \quad 0 & 0 & \dots & 0 & 1 \end{array}$$

Letting c denote the nth bit of $\tau(\tilde{x})$, we have

$$b_1 \oplus \cdots \oplus b_n = \begin{cases} c \oplus b_n \oplus 1 & \text{if } n+1 \text{ is even and } \left\lfloor \frac{n+1}{2} \right\rfloor \text{ is odd} \\ c \oplus b_n & \text{if } n+1 \text{ is even and } \left\lfloor \frac{n+1}{2} \right\rfloor \text{ is even} \\ c \oplus 1 & \text{if } n+1 \text{ is odd and } \left\lfloor \frac{n+1}{2} \right\rfloor \text{ is odd} \\ c & \text{if } n+1 \text{ is odd and } \left\lfloor \frac{n+1}{2} \right\rfloor \text{ is even,} \end{cases}$$

where $\oplus$ denotes exclusive-or. Here the parity of $n+1$ determines parity of the b_n's in the nth position while the parity of $\lfloor \frac{n+1}{2} \rfloor$ determines whether there is a carry-out from the $(n+1)$st position to the $(n-1)st$ position. So, by the assumption on $\{C_n\}$, we may obtain a family $\{D_n\}$ of constant-depth poly-sized circuits such that $D_n(b_1 \dots b_n) = b_1 \oplus \cdots \oplus b_n$, contrary to the fact that PARITY $\notin \mathsf{AC}^0$.

We have ruled out AC^0-approximation of Takagi's function. Now consider the following function:

$$T(x) = \sum_{k=0}^{\infty} \frac{1}{2^{2^k-1}} \langle\!\langle 2^{2^k-1} x \rangle\!\rangle.$$

Suppose $x, \tilde{x} \in [0,1]$ where $\tilde{x} = \sum_{i=1}^{2n+2} b_i 2^{-i}$ and $|x - \tilde{x}| \le 2^{-(2n+2)}$. Here we have

$$\begin{aligned} |T(x) - T(\tilde{x})| &= \left| \sum_{k=0}^{\infty} \frac{1}{2^{2^k-1}} \langle\!\langle 2^{2^k-1} x \rangle\!\rangle - \sum_{k=0}^{\infty} \frac{1}{2^{2^k-1}} \langle\!\langle 2^{2^k-1} \tilde{x} \rangle\!\rangle \right| \\ &= \left| \sum_{k=0}^{\infty} \frac{1}{2^{2^k-1}} \langle\!\langle 2^{2^k-1} x \rangle\!\rangle - \sum_{k=0}^{\lceil \log(n+1) \rceil + 1} \frac{1}{2^{2^k-1}} \langle\!\langle 2^{2^k-1} \tilde{x} \rangle\!\rangle \right| \\ &\le \sum_{k=0}^{\lceil \log(n+1) \rceil + 1} \frac{1}{2^{2^k-1}} \left| \langle\!\langle 2^{2^k-1} x \rangle\!\rangle - \langle\!\langle 2^{2^k-1} \tilde{x} \rangle\!\rangle \right| + \sum_{k=\lceil \log(n+1) \rceil + 2}^{\infty} \frac{1}{2^{2^k-1}} \langle\!\langle 2^{2^k-1} x \rangle\!\rangle \\ &\le \sum_{k=0}^{\lceil \log(n+1) \rceil + 1} \frac{1}{2^{2^k-1}} \cdot \frac{1}{2^{2n-(2^k-1)+2}} + \sum_{k=\lceil \log(n+1) \rceil + 2}^{\infty} \frac{1}{2^{2^k}} \\ &\le \frac{\lceil \log(n+1) \rceil + 2}{2^{2n+2}} + \frac{1}{2^{2n+2}} \le \frac{1}{2^n} \end{aligned}$$

So $T(\tilde{x})$ provides a $2^{-O(n)}$ approximation to $T(x)$. $T(\tilde{x})$ is obtained by adding $\lceil \log(n+1) \rceil + 2$ of numbers of $2n+2$ bits where each number can be obtained in AC^0. Since such sums may be performed in AC^0 (see, e.g., [Vol99]). In conclusion, we can $2^{-O(n)}$-approximate T in AC^0. Here we can use an immediate adaptation to T of similar proofs (e.g., [Bil82]) for τ. The continuity of T follows by the Weierstrass M-test (or from the fact that it can be approximated.) Finally, letting $\varphi_k(x)$ denote $\frac{1}{2^{2^k-1}} \langle\!\langle 2^{2^k-1} x \rangle\!\rangle$, if T were differentiable at $x \in [0,1]$, we would have $T'(x) = \sum_{k=0}^{\infty} \varphi_k'(x)$. However, this sum diverges since for all k, $|\varphi_k'(x)| = 1$. This intuition can be formalized to show that T is nowhere differentiable.

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THE INFINITE LOOP OPERATION AND THE AXIOM OF DEPENDENT CHOICE

TAKAYUKI KIHARA

Recently, Brattka [1] introduced the notion of infinite loop operation $(-)^\infty$ on Weihrauch problems. Applying Yoshimura's unpublished theorem [3], one can see that a Weihrauch problem F is closed under the infinite loop operation (the inverse limit) if and only if F-relative realizability validates the axiom of *dependent choice* DC. Therefore, it is an important problem to investigate which Weihrauch problems are closed under the infinite loop operation. One question proposed in the recent Dagstuhl seminar 25131 "Weihrauch Complexity: Structuring the Realm of Non-Computability (Mar. 23–28, 2025)," was to determine the strength of the infinite loop closure of LLPO_k (all-or-counique choice on k).

Definition 1. LLPO_k:

- Input: (a code of) $(n_i)_{i<k} \in \mathbb{S}^k$ such that $n_i = 1$ for at most one i.
- Output: $i < k$ such that $n_i = 0$.

Here, $\mathbb{S}$ is the Sierpiński space. The principle LLPO_k is first introduced by Richman [2] in the context of constructive mathematics. This principle is also called ACC_k (all-or-counique choice for k-elements) in the context of Weihrauch degrees.

Question 1. *Is* $\mathsf{LLPO}_{k+1}^{\infty\infty\infty\cdots} <_W \mathsf{LLPO}_k^{\infty\infty\infty\cdots}$*?*

We solve this problem by showing that $\mathsf{LLPO}_k^{\infty\infty\infty\cdots}$ is equivalent to DNR_k. Here, DNR stands for a **d**iagonally **n**on-**r**ecursive function[1].

Definition 2. DNR_k:

- Input: (a code of) a partial function $f :\subseteq \mathbb{N} \to k$.
- Output: $g \in k^{\mathbb{N}}$ such that $g(n) \neq f(n)$ for any $n \in \mathrm{dom}(f)$.

Theorem 3. $\mathsf{LLPO}_k^{\infty\infty} \equiv_{\mathsf{W}} \mathsf{DNR}_k^{\infty} \equiv_{\mathsf{W}} \mathsf{DNR}_k$.

This has the following conclusion, for example.

Corollary 4. $\mathsf{IZF} + \mathsf{DC} + \mathsf{MP} + \mathsf{LLPO}_{k+1}$ *does not imply* LLPO_k.

Here, IZF stands for intuitionistic ZF set theory, and MP stands for Markov's principle.

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[1] In the original context, this refers to a total function that completely avoids a universal partial computable function, but in the Weihrauch context, we consider a relativized version of this, which corresponds to the parallelization of LLPO.

Relational Reasoning for Verified Reiterative Implementations of Multivalued Real Computations

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We recall relational program logic [Fra83, MHRVM19] for ordinary programming languages over intervals with discrete, multi-precision endpoints, to verify the correctness of reiterative implementations of exact real number computations, such as iRRAM [Mül00]. The C++ implementation of iRRAM introduces a so-called multivalue cache, consisting of shared stores preserved and accessed throughout reiterations. It is used to achieve consistency in multivalued computation: when a computation is repeated with higher precision, the computation path, including IO values, must remain consistent. Such consistency is a crucial property of exact real number computation. It justifies hiding internal representations and reiterations, allowing users to reason about real numbers based on the familiar structure of abstract real numbers [BCZ22, PBC$^+$24, BPS24]. We show that reiteration consistency can be encoded as a relational property, and propose relational correctness, together with reasoning invariant over the multivalue cache, as a plausible approach to verifying whether implementations truly achieve this consistency.

We further show how our framework for reasoning about implementation correctness can be extended to arbitrary continuous data types (e.g., [LLPZ19, HP20]) and their representations in the sense of computable analysis [Wei00]. We claim that this is particularly useful when a complex data type requires native low-level support for performance reasons.

It is addressed in [PT23] that reiterative implementations of real computations, in general, fail to provide modularity when, for example, c_1 and c_2 interfere due to a shared multi-valued cache. In such cases, the limit behavior of $c_1; c_2$ may not be identical to the composition of the two. We overcome this problem by enforcing strong physical separation in the cache to allow program composition. Easing this condition into weaker logical separation remains a main direction for future work.

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RANDOMNESS WITH RESPECT TO C.E. SEMIMEASURES

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Martin-Löf randomness with respect to computable measures is typically defined through the concept of tests. It also has robust characterizations via martingales and complexity. Recently, randomness with respect to c.e. semimeasures has begun to attract attention. In this talk, we propose several definitions of randomness, particularly focusing on those based on complexity, and clarify the hierarchy among them.

Bienvenue et al. [2] is probably the first study in this area, which focused on *randomness preservation* and *no-randomness-from-nothing*. These theorems roughly state that randomness with respect to a computable measure is preserved under computable mappings defined almost everywhere. This preservation does not hold for a partial computable mapping, whose push-forward measure is a c.e. semimeasure. Bienvenue et al. [2] showed that there exist two partial computable functions with the same push-forward measure but differing images of random sequences.

A related paper by Barmpalias and Shen [1] explores a notion of randomness for c.e. semimeasures different from the one proposed by Bienvenu et al. [2]. Under additional conditions, they establish a result analogous to no-randomness-from-nothing.

We clarify the situation by defining several concepts of randomness and examining the hierarchical structure among them. Our findings are best viewed in comparison with the hierarchy of randomness concepts considered in the context of partial randomness. Hudelson's doctoral thesis [3] serves as a useful survey on this topic.

Now we define some randomness notions with respect to c.e. semimeasures μ. We consider the following four notions.

(I) KA-f-complexity.
(II) Strong K-f-complexity.
(III) K-f-complexity.
(IV) f-ML-randomness.

The former three notions have been appeared in Hudelson [3, Page 13]. The notion (IV) has been defined in Bienvenue et al. [2]. The notion (I) with additional conditions are considered in Barmpalias and Shen [1].

Let $f : 2^{<\omega} \to [0, \infty]$ be a function. We intend to let $f(\sigma) = -\log \mu(\sigma)$ for a c.e. semimeasure μ. Thus, f is usually a upper semi-computable function.

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Definition 1 (complexity)**.** A sequence $X \in 2^\omega$ is called *KA-f-complex* if

$$K(X \upharpoonright n) > f(X \upharpoonright n) - O(1).$$

A sequence $X \in 2^\omega$ is called *strongly K-f-complex* if

$$K(X \upharpoonright n) - f(X \upharpoonright n) \to \infty \text{ as } n \to \infty.$$

A sequence $X \in 2^\omega$ is called *K-f-complex* if

$$K(X \upharpoonright n) > f(X \upharpoonright n) - O(1).$$

For a definition of ML-randomness for a c.e. semimeasure, we use a sequence of strings rather than a c.e. open set.

Definition 2 (ML-randomness)**.** A *f-ML-test* is a sequence of uniformly c.e. sets $S_n \subseteq 2^{<\omega}$ such that $\sum_{\sigma \in S_n} 2^{-f(\sigma)} \leq 2^{-n}$ for all $n \in \omega$. A sequence $X \in 2^\omega$ is called *f-ML-random* if $X \notin \bigcap_n [S_n]$ for each f-ML-test $(S_n)_n$.

Some researchers have called this notion dwt-f-randomness.

The main goal is to show the following implications:

$$\text{(I)} \Rightarrow \text{(II)} \Rightarrow \text{(III)} \Rightarrow \text{(IV)}.$$

Each implication is strict.

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CONSTRUCTIBLE FAILURES OF ERDŐS-VOLKMANN FOR RINGS

LINUS RICHTER

1. Introduction

Set-theoretical axioms and the structure of the real line are deeply intertwined. The investigation of their relationship pitches the definable structure of sets of real numbers against the behaviour of non-constructive existence axioms. The former camp is classically represented by the Borel sets; the latter by the Axiom of Choice. Let P be a property of sets of reals. The following pattern has emerged frequently:

(1)	ZFC	P holds for every Borel set
(2)	$\mathsf{ZFC}+\mathsf{CH}$	P fails for some non-Borel set
(3)	$\mathsf{ZF}+\mathsf{DC}+\mathsf{AD}$	P holds for every set
(4)	$\mathsf{ZFC}+V{=}L$	P fails for some co-analytic set

Examples of properties following this pattern include the perfect set property and the property of Baire, Marstrand projection, and the Solecki dichotomy [2, 6, 13, 16, 12]. Normally, (3) follows from (1) by expressing P game-theoretically. Then (1) follows from Borel determinacy, and (3) follows from AD; (2) is often a diagonalisation argument. But what is the *least* complexity class for which (2) holds, assuming ZF? The complexity of CH-constructed "counterexamples" in (2) is not finely calibrated. This leads to the following research question:

"Find the least Γ such that there consistently exists $X \in \Gamma$ for which P fails."

Z. Vidnyánszky [14] has generalised a method first used by P. Erdős, K. Kunen and R. D. Mauldin [4], and by A. Miller [11] to recursively construct $\mathbf{\Pi}^1_1$ sets of reals, assuming $V{=}L$; this is a fruitful tool to establish item (4) in the table above. Notably, Vidnyánszky's theorem lends itself to applications in fractal geometry. Via Lutz' and Lutz' point-to-set-principle [7], algorithmic randomness and effective dimensions of reals can be used to construct sets of pathological fractal properties. hence, we aim to contribute to the general research programme of structurally classifying the proof-theoretic strength of ZF for regularity properties of sets of reals in the context of fractal geometry. Here, we focus on the Erdős-Volkmann-problem for rings.

2. Erdős-Volkmann for Rings

The Erdős-Volkmann-problem is a ring-theoretical problem originally posed by B. Volkmann [15] and partially resolved by P. Erdős and B. Volkmann [5], which contrasts the algebraic structure of subrings of $\mathbb{R}$ with their geometric structure. To state their work, let Γ denote a pointclass (e.g. Borel $= \mathbf{\Delta}^1_1$ or analytic $= \mathbf{\Sigma}^1_1$). We denote by $P_{\mathsf{ring}}(\Gamma)$ the property:

If $B \subseteq \mathbb{R}$ is a proper subring of $\mathbb{R}$ and $B \in \Gamma$, then $\dim_H(B) = 0$ or $B = \mathbb{R}$.

In 1966, Erdős and Volkmann showed that, for Borel sets, the notion of subgroup is too weak to preserve high geometric structure as characterised by $P_{\mathsf{ring}}(\mathbf{\Delta}^1_1)$ [5]:

Theorem 2.1 (ZF, Erdős-Volkmann-Theorem)**.** *For every $s \in [0,1]$ there exists a Borel subgroup $G \leq \mathbb{R}$ such that $\dim_H(G) = s$. In other words, $P_{\mathsf{group}}(\mathbf{\Delta}^1_1)$ is false.*

For sub*rings*, the situation is different, as was shown by G. Edgar and C. Miller [3] and independently Bourgain [1]:

Theorem 2.2 (ZF, Edgar-Miller-Bourgain-Theorem)**.** *If $B \subseteq \mathbb{R}$ is a proper subring of $\mathbb{R}$ that is $\mathbf{\Sigma}^1_1$ then $\dim_H(B) = 0$ or $B = \mathbb{R}$. In other words, $P_{\mathsf{ring}}(\mathbf{\Sigma}^1_1)$ is true.*

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R. O. Davies contributed item (2) by constructing subrings of arbitrary dimension. By the Edgar-Miller-Bourgain-Theorem, these subrings are not $\mathbf{\Sigma}^1_1$. However, Davies' proof is unpublished (cf. [9, p. 167]). In 2016, Mauldin provided the details by "completing an attack first discovered by Roy Davies" [10] and showed:

Theorem 2.3 (CH, Davies-Mauldin-Theorem)**.** *For every $s \in (0,1)$ there exists a proper subring $B \subseteq \mathbb{R}$ for which $\dim_H(B) = s$.*

This leaves items (3) and (4) open—this is one interpretation of the Erdős-Volkmann-ring-problem.

3. Our Contribution

We report on ongoing work to establish item (4) for the Erdős-Volkmann ring problem. Precisely, we aim to construct, for every $s \in (0,1)$, a proper $\mathbf{\Pi}^1_1$-subring $A \subseteq \mathbb{R}$ of Hausdorff dimension s. Such a result would prove that ZFC is not powerful enough to prove $P_{\mathsf{ring}}(\mathbf{\Pi}^1_1)$. The difficulty lies in the Davies-Mauldin-proof, which cannot be trivially effectivised to make use of Vidnyánszky's theorem alongside the point-to-set-principle. We explain these difficulties, which are both algebraic and algorithmic, and motivate possible workarounds. For instance, a classification of sets of bounded Hausdorff dimension, in the style of Marcone and Valenti [8] could be useful in transferring the Davies-Mauldin proof into Vidnyánszky's framework.

Further, it is our hope to isolate a criterion which, especially in the context of fractal geometry, gives a uniform description of those properties P for which item (4) holds. The connection between classical Hausdorff dimension and algorithmic randomness appears fruitful to provide such a characterisation. This would be of additional interest as the Erdős-Volkmann problem is also influenced by additional algebraic structure.

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A Verified Power-Series Method for Multivariate IVPs*

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We formalize and verify an algorithm for rigorously solving initial value problems (IVPs) for systems of multivariate analytic ordinary differential equations (ODEs) based on ideas from computable analysis, within the Rocq (formerly Coq) proof assistant. The goal is to provide verified computational solutions with explicit error control based on constructive implementations of real numbers in the proof assistant. More precisely, we consider IVPs for systems of (autonomous) first-order analytic ODEs of the form

$$\dot{\vec{y}}(t) = \vec{f}(\vec{y}(t)), \quad \vec{y}(0) = \vec{y_0} \tag{1}$$

with $\vec{f} : U \subseteq \mathbb{R}^d \to \mathbb{R}^d$.

In the context of computable analysis often a method based on computing the *infinite* Taylor series expansion at a point is used [BKM16, KST18, SSTZ21] for analytic IVPs. This method is efficient in this framework as error bounds decrease exponentially in the order of the expansion and therefore (in contrast to e.g. the Euler method or Runge-Kutta methods) allows feasible computation for high precision [Mül95]. The method is based on the ODE version of the Cauchy-Kovalevskaya theorem and its classical proof (see e.g. [Tes12, Chapter 4]). The main idea is to compute the power series of the solution by recursively determining higher-order derivatives, that is, we define a sequence $F^{[n]} : \mathbb{R}^d \to \mathbb{R}^d$ of multivariate analytic functions by

$$F^{[0]}(\vec{x}) = \vec{x}, \quad F^{[n+1]}(\vec{x}) = \vec{f}(\vec{x}) \cdot \mathrm{D}\big(F^{[n]}\big)(\vec{x}),$$

where D denotes the Jacobian. The resulting infinite series $\vec{y}(t) = \sum_{i=0}^{\infty} \frac{1}{n!} \vec{F}^{[n]}(y_0) t^n$ converges effectively to the unique solution of (1). Further, rigorous error bounds can be computed explicitly from simple bounds on the right-hand side function f. Note that the resulting power series is never finite except for trivial cases.

Previous work [PT24] by Park and the author presented a formalization of this method in Rocq as part of the certified exact real computation library cAERN [KPT24]. However, the implementation only supports *one-dimensional polynomial* ODEs, and thus has limited practical applications. Furthermore, cAERN's main purpose is the extraction of efficient and certified programs for the exact real computation library AERN and it thus does not support direct computation within the proof assistant. Although extracted programs are typically more efficient than computing inside of Rocq, this feature can still be useful, especially since it allows to include the results of computations inside formal proofs.

In this talk, we present a new Rocq formalization[1] which is applicable to systems of *arbitrary* dimension and to generic analytic right-hand side functions. Furthermore, instead of using a concrete type for real numbers and functions, we use type classes and setoids to abstractly specify which kind of properties are needed. The type-classes can be instantiated with any type satisfying these properties, making the formalization flexible and easily adaptable to different formalizations of constructive real numbers in Rocq, including the implementation in the Rocq standard library. The development is modular, giving precise control over the properties required in different parts of the formalization. This makes it straightforward to

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[1]`https://github.com/holgerthies/taylor_rocqs`

adapt the solver not only to exact real-number types but also to discrete numeric types such as rationals or arbitrary-precision floating-point numbers, allowing to compute exact approximations with rigorous error bounds. We further demonstrate how the results can be extended to interval arithmetic, resulting in a verified solver that efficiently computes rigorous interval enclosures for ODE trajectories.

To illustrate the practicality of our formalization, we instantiate and benchmark it using multiple concrete constructive real number frameworks, including the standard library's Cauchy reals, an implementation from the CoRN library [CFGW04], and an efficient interval version based on the CoqInterval library [MDM16], and test the implementation on several classical and practically relevant example ODEs.

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