Non-deterministic analogue computations with ODEs: Towards a characterisation of NP

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In 1941, Claude Shannon introduced in [12] a model of analogue computation, the GPAC (general purpose analogue computer). This was proposed as a mathematical model of (mechanical) analogue computers of that time, namely differential analysers. In this model, we can compose basic blocs (constants, additions, multiplications, integrators) into circuits (feedbacks are allowed) in order to compute functions. A GPAC can also be thought of as computing solutions of polynomial initial value problems (PIVPs), a particular class of ordinary differential equations. A characterisation in that direction was obtained in [5]. In terms of computability, it was first thought to be a less powerful model than the Turing machine model, but it is now regarded as equivalent, as long as we use a coherent definition of computation for this model, following [3]. In practice, from strong closure properties, it can be considered as a model of any (old or modern) analogue computing device.

The question of complexity was addressed only recently by [11,4], who proved that the time complexity of the computation corresponded to the length of the curve of the solution of the ODE associated with the GPAC. More precisely, a real function is computable if and only if it is computable by a GPAC of polynomial length [11,4]. Very recently, it was also proved that space corresponds to precision [1,2]. All these results, providing analogue characterisations of P, and PSPACE at the end, were obtained by proving both that Turing machines computations could be simulated by various classes of ordinary differential equations in one direction, and arguments from computable analysis in the other direction. In particular, the statement about the time complexity is highly based on the fact that there is a way to solve ordinary differential equations in a time that is polynomial in the length of the solution.

In the more traditional computable analysis approach, the complexity of solving ODEs has been investigated by many works [8,9,10,6]. Concerning analytic functions, the work by [13,7] showed that we could compute solutions of slightly more general problems than PIVPs, namely IVPs with analytic right-hand sides. The approach is based on parametrised complexity, where some parameters are assumed to be encoded in unary, while others have their usual representation in binary [13,7]. There, the length of the curve was not used, but instead specific parametric representations that would give the information needed to make evaluation and ODE-solving polynomial-time computable. These representations each consist of a way to represent the analytic function itself (by an approximation of the function or by the coefficients of the associated series) and of some bounds on natural quantities, which can be interpreted as radii of convergence, that make the operations that we want to perform polynomial-time computable. As noticed by both authors, the (parametrised) algorithms that they proposed have similarities, even if both admit that this requires more investigation.

One first outcome of this work is to explain the relations between both. More generally, the goal of the present work is motivated by extending previous statements about analogue computations, in order to deal with non-determinism: can we provide a characterisation of the NP class, using ordinary differential equations in the spirit of the above analogue characterisations of P and PSPACE? Concretely, an idea is that by adding a discrete parameter in a discrete-time system, one can simulate non-determinism: a non-deterministic computation succeeds if the computation succeeds for some value of the parameter, which means performing an existential quantification on the parameter. This provides one of the directions. Conversely, the idea is that by using approximation theory by analytic functions and even polynomials, one can add such a parameter in a continuous-time system, and under some conditions, the good behaviour and complexity are preserved. If this idea is simple, the deep key point is to identify the class of allowed functions or dynamics that may be authorised in the quantification, in order to remain in the class NP.

The proposed work identifies a suitable class of such dynamics. We unify the parameters from [13] (the radius of convergence over the reals and a bound for the first representation, the size of a subdomain over the complex and another bound for the second one) and the length from [11] to get a better understanding of the quantity that we have to bound to guarantee polynomial-time computability. As another side effect, we also identify the role played by absolute continuity in involved reasoning and representations.

This work corresponds to the master's thesis of the second author (in alphabetical order). His report will be available on: `https://www.lix.polytechnique.fr/~bournez/CCA2025/` as soon as possible. A preliminary version (in French) is already available at this url.

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