COMPUTABLE BASES

VASCO BRATTKA AND EMMANUEL RAUZY

Abstract. In computable analysis typically topological spaces with countable bases are considered. The Theorem of Kreitz-Weihrauch implies that the subbase representation of a second-countable T_0 space is admissible with respect to the topology that the subbase generates. We consider generalizations of this setting to bases that are representable, but not necessarily countable. We introduce the notions of a computable presubbase and a computable prebase. We prove a generalization of the Theorem of Kreitz-Weihrauch for the presubbase representation that shows that any such representation is admissible with respect to the topology generated by compact intersections of the presubbase elements. For computable prebases we obtain representations that are admissible with respect to the sequentialization of the topology that they generate. These concepts provide a natural way to investigate many topological spaces that are studied in computable analysis.

1. Summary

We start with defining the concept of a presubbase.

Definition 1.1 (Presubbase). Let X be a set. We call a family $(B_y)_{y\in Y}$ a *presubbase* for X, if Y is a represented space and its *transpose*

$$B^{\mathsf{T}} : X \to \mathcal{O}(Y), x \mapsto \{y \in Y : x \in B_y\}$$

is well-defined and injective.

Injectivity of B^{T} implies that $(B_y)_{y\in Y}$ is a subbase of some T_0 topology on X. We note that every countable subbase $B : \mathbb{N} \to \mathcal{O}(X)$ of a T_0 topology is a particular instance of a presubbase, as $B^{\mathsf{T}} : X \to \mathcal{O}(\mathbb{N})$ is always well-defined. Hence, the following definition generalizes the concept of a subbase representation as it is known in computable analysis [Wei00].

Definition 1.2 (Presubbase representation). Let $(B_y)_{y\in Y}$ be a presubbase of a set X. We define the *presubbase representation* $\delta^B :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ by

$$\delta^B(p) = x :\iff \delta_{\mathcal{O}(Y)}(p) = \{y \in Y : x \in B_y\}$$

for all $p \in \mathbb{N}^{\mathbb{N}}$ and $x \in X$.

The reason that we speak about a *pre*subbase representation in this general situation and not about a subbase representation is that δ^B is not necessarily admissible with respect to the topology generated by $(B_y)_{y\in Y}$. However, it is admissible with respect to a closely related topology generated by compact intersections of the sets B_y, as shown in the next theorem. We call a represented space X a *computable Kolmogorov space* if its neighborhood map $\mathcal{U} : X \to \mathcal{O}\mathcal{O}(X), x \mapsto \{U \in \mathcal{O}(X) : x \in U\}$ is a computable embedding. This is the natural effectivization of the T_0 property (that is sometimes called *computable admissibility*) [Sch02, Sch21].

Theorem 1.3 (Presubbase theorem). *Let $(B_y)_{y\in Y}$ be a presubbase of a set X. Then (X, δ^B) is a computable Kolmogorov space and δ^B is admissible with respect to the topology τ on X with the base sets X and $\bigcap_{y\in K} B_y$ for every compact $K \subseteq Y$.*

We note that this result generalizes the Theorem of Kreitz-Weihrauch [KW85] as for countable subbases $B : \mathbb{N} \to \mathcal{O}(X)$ the compact subsets $K \subseteq \mathbb{N}$ are exactly the finite subsets and hence the topology generated by X and $\bigcap_{n\in K} B_n$ for compact $K \subseteq \mathbb{N}$ is exactly the same topology as the topology generated by the subbase B itself. We can now define the notion of a computable presubbase.

Definition 1.4 (Computable presubbase)**.** Let X and Y be represented spaces. Then $B : Y \to \mathcal{O}(X)$ is called a *computable presubbase* of X if the transpose

$$B^{\mathsf{T}} : X \to \mathcal{O}(Y), x \mapsto \{y \in Y : x \in B_y\}$$

is well-defined and a computable embedding.

Obviously, a presubbase B of X is a computable presubbase of X if and only if the representation of X is computably equivalent to δ^B. In analogy to the countable case [BR25] we can now define the concept of a computable base.

Definition 1.5 (Computable prebase)**.** Let X a represented space with a computable presubbase $B : Y \to \mathcal{O}(X)$. Then B is called a *computable prebase* of X, if there is a computable $R : \mathcal{K}_-(Y) \rightrightarrows \mathcal{A}_+(Y)$ such that $\bigcap_{y\in K} B_y = \bigcup_{y\in A} B_y$ for every $K \in \mathcal{K}_-(Y)$ and $A \in R(K)$ and $X = \bigcup_{y\in Y} B_y$. We call B a *computable base* of X if B is actually a base of X

By Theorem 1.3 computable prebases characterize the topology of their spaces up to sequentialization, which one can see using results of Schröder [Sch02]. In fact, every computable Kolmogorov space has a computable base, namely the identity. Altogether, we obtain the following characterization of computable Kolmogorov spaces in terms of their bases.

Theorem 1.6 (Computable Kolmogorov spaces and bases)**.** *Let X be a represented space X. Then the following are pairwise equivalent:*

(1) X is a computable Kolmogorov space,
(2) X has a computable presubbase,
(3) X has a computable prebase,
(4) X has a computable base,
(5) $\mathrm{id} : \mathcal{O}(X) \to \mathcal{O}(X)$ is a computable base of X.

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Faculty of Computer Science, Universität der Bundeswehr München, Germany and Dept. of Mathematics & Appl. Math. University of Cape Town, South Africa[1]
E-mail address: `Vasco.Brattka@cca-net.de`

Faculty of Computer Science, Universität der Bundeswehr München, Germany[2]
E-mail address: `emmanuel.rauzy.14@normalesup.org`

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