Subrecursive degrees of difference representations of irrational numbers

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Abstract. We explore the degree structure of representations of irrational numbers, induced by subrecursive reducibility. The representation R_1 is subrecursive in the representation R_2, denoted $R_1 \preceq_S R_2$, if there exists a Turing operator, which transforms any R_2-representation of an irrational α into some R_1-representation of the same α. The operator is not allowed to use unbounded minimization and it must work uniformly in α.

For example, we have $\mathcal{C} \prec_S \mathcal{E}_b \prec_S \mathcal{D} \prec_S [\,]$, where $\mathcal{C}$ is the representation by Cauchy sequences with fixed convergence rate, $\mathcal{E}_b$ is the representation by base-b expansions, $\mathcal{D}$ is the representation by Dedekind cuts and $[\,]$ is the representation by continued fractions.

This degree structure was formally introduced in [1] quite recently, but the study of concrete representations in this context can be traced back to the foundational papers [7,8] and since then it has been constantly evolving, see for example [2,3,5,6].

In this talk we will present a novel idea to define new representations based on difference from rational numbers. Somewhat surprisingly, this idea produces numerous new subrecursive degrees.

Let R be a representation of the real numbers, such that any $\alpha \in [0,1]$ has a unique representation R^α. We assume $R^\alpha : A \to B$, where A, B are fixed primitive recursively encoded countable sets. For any $\alpha \in (0,1) \setminus \mathbb{Q}$ we define:

$$\mathcal{D}\mathrm{iff}_R^\alpha(q) \;=\; \mu a \in A[R^\alpha(a) \neq R^q(a)],$$

where the minimum is regarded with respect to the coding of A.

We will need some assumptions, so that $\mathcal{D}\mathrm{iff}_R$ is a representation in the sense of [1], that is $\mathcal{D}\mathrm{iff}_R^\alpha$ is uniformly computably equivalent to the Dedekind cut $\mathcal{D}^\alpha$:

1. The restriction of R to rational numbers, $R : \mathbb{Q} \times A \to B$, must be computable.
2. There exists a binary computable function P, such that whenever $\overline{a}$ is a finite list of elements of A and $\overline{b}$ is a finite list of elements of B with the same length, $P(\overline{a}, \overline{b})$ produces a rational number q, which satisfies $R^q(a_i) = b_i$ for all i.

The first assumption implies that we can compute $\mathcal{D}\mathrm{iff}_R^\alpha$ using oracle R^α (in general, unbounded search is needed for that).

The second assumption allows computation of $R^\alpha(a)$ by the following algorithm:

1. Compute all elements $a_0, \ldots, a_k = a$ of A with code at most the code of a. Assume that we have inductively computed $R^\alpha(a_i)$ for $i < k$.

2. For each $b_k \in B$:

   Compute $q = P(\overline{a}, \overline{b})$, where $b_i = R^\alpha(a_i)$ for $i < k$.

   If $\mathcal{D}\mathrm{iff}_R^\alpha(q) \neq a$, then return output $R^\alpha(a) = b_k$.

This is a computable reduction from $\mathcal{D}\mathrm{iff}_R^\alpha$ to R^α, therefore $\mathcal{D}\mathrm{iff}_R^\alpha$ is computably equivalent to R^α. So whenever R is a representation in the sense of [1], $\mathcal{D}\mathrm{iff}_R$ is also a representation.

Note that in case B is finite and P is primitive recursive no unbounded search is needed in the above algorithm. But if B is infinite, $\mathcal{D}\mathrm{iff}_R$ and R might even turn out to be subrecursively incomparable.

As an example, let us consider the representation $\mathcal{D}\mathrm{iff}_b$, which corresponds to the base-b expansion $\mathcal{E}_b$. The base-b expansions of numbers of the form $\frac{m}{b^n}$ are assumed to end in zeros.

The base-b sum approximation from below $\hat{A}_b^\alpha$ of $\alpha \in [0,1]$ is defined as $\hat{A}_b^\alpha(n) = \mathtt{D}_n b^{-k_n}$, where $\mathtt{D}_n \neq 0$ and $\alpha = \sum_{n=0}^{\infty} \mathtt{D}_n \cdot b^{-k_n}$. The base-$b$ sum approximation from above $\check{A}_b^\alpha$ is defined symmetrically, so that $\check{A}_b^\alpha(n) = \hat{A}_b^{1-\alpha}(n)$. The base-$b$ sum approximations of rational numbers with finite base-b expansion are assumed to end in zeros. The general sum approximation from below (from above) of $\alpha \in [0,1]$ is defined as $\hat{G}^\alpha(b,n) = \hat{A}_b^\alpha(n)$ ($\check{G}^\alpha(b,n) = \check{A}_b^\alpha(n)$).

In the paper [4], it is shown that $\hat{G} \prec_S \mathcal{D}\mathrm{iff}_b \prec_S [\,]$.

Our aim here is to present some new results for the representation $\mathcal{D}\mathrm{iff}_{b\uparrow}$ corresponding to base-b sum approximations from below (symmetric results hold for $\mathcal{D}\mathrm{iff}_{b\downarrow}$).

Theorem 1. $\mathcal{E}_b \preccurlyeq_S \mathcal{D}\mathrm{iff}_{b\uparrow}$ and $\mathcal{D} \preccurlyeq_S \mathcal{D}\mathrm{iff}_{b\uparrow}$.

Theorem 2. $\mathcal{D}\mathrm{iff}_{b\uparrow} \not\preccurlyeq_S \hat{G}$ and $\mathcal{D}\mathrm{iff}_{b\uparrow} \not\preccurlyeq_S \check{G}$.

Theorem 3. $\hat{A}_b \not\preccurlyeq_S \mathcal{D}\mathrm{iff}_{b\uparrow}$ and $\check{A}_b \not\preccurlyeq_S \mathcal{D}\mathrm{iff}_{b\uparrow}$.

Keywords: representations of irrational numbers, subrecursive reducibility, difference representations

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