THE INFINITE LOOP OPERATION AND THE AXIOM OF DEPENDENT CHOICE

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Recently, Brattka [1] introduced the notion of infinite loop operation $(-)^\infty$ on Weihrauch problems. Applying Yoshimura's unpublished theorem [3], one can see that a Weihrauch problem F is closed under the infinite loop operation (the inverse limit) if and only if F-relative realizability validates the axiom of *dependent choice* DC. Therefore, it is an important problem to investigate which Weihrauch problems are closed under the infinite loop operation. One question proposed in the recent Dagstuhl seminar 25131 "Weihrauch Complexity: Structuring the Realm of Non-Computability (Mar. 23–28, 2025)," was to determine the strength of the infinite loop closure of LLPO_k (all-or-counique choice on k).

Definition 1. LLPO_k:

- Input: (a code of) $(n_i)_{i<k} \in \mathbb{S}^k$ such that $n_i = 1$ for at most one i.
- Output: $i < k$ such that $n_i = 0$.

Here, $\mathbb{S}$ is the Sierpiński space. The principle LLPO_k is first introduced by Richman [2] in the context of constructive mathematics. This principle is also called ACC_k (all-or-counique choice for k-elements) in the context of Weihrauch degrees.

Question 1. *Is* $\mathsf{LLPO}_{k+1}^{\infty\infty\infty\cdots} <_W \mathsf{LLPO}_k^{\infty\infty\infty\cdots}$*?*

We solve this problem by showing that $\mathsf{LLPO}_k^{\infty\infty\infty\cdots}$ is equivalent to DNR_k. Here, DNR stands for a **d**iagonally **n**on-**r**ecursive function[1].

Definition 2. DNR_k:

- Input: (a code of) a partial function $f\colon\subseteq \mathbb{N} \to k$.
- Output: $g \in k^{\mathbb{N}}$ such that $g(n) \neq f(n)$ for any $n \in \mathrm{dom}(f)$.

Theorem 3. $\mathsf{LLPO}_k^{\infty\infty} \equiv_{\mathsf{W}} \mathsf{DNR}_k^{\infty} \equiv_{\mathsf{W}} \mathsf{DNR}_k$.

This has the following conclusion, for example.

Corollary 4. $\mathsf{IZF} + \mathsf{DC} + \mathsf{MP} + \mathsf{LLPO}_{k+1}$ *does not imply* LLPO_k*.*

Here, IZF stands for intuitionistic ZF set theory, and MP stands for Markov's principle.

References

[1] Vasco Brattka. Loops, inverse limits and non-determinism. arXiv:2501.17734, 2025.
[2] Fred Richman. Polynomials and linear transformations. *Linear Algebra Appl.*, 131:131–137, 1990.
[3] Kazuto Yoshimura. General treatment of non-standard realizabilities. submitted, 2016.

[1]In the original context, this refers to a total function that completely avoids a universal partial computable function, but in the Weihrauch context, we consider a relativized version of this, which corresponds to the parallelization of LLPO.