RANDOMNESS WITH RESPECT TO C.E. SEMIMEASURES

KENSHI MIYABE

Martin-Löf randomness with respect to computable measures is typically defined through the concept of tests. It also has robust characterizations via martingales and complexity. Recently, randomness with respect to c.e. semimeasures has begun to attract attention. In this talk, we propose several definitions of randomness, particularly focusing on those based on complexity, and clarify the hierarchy among them.

Bienvenue et al. [2] is probably the first study in this area, which focused on *randomness preservation* and *no-randomness-from-nothing*. These theorems roughly state that randomness with respect to a computable measure is preserved under computable mappings defined almost everywhere. This preservation does not hold for a partial computable mapping, whose push-forward measure is a c.e. semimeasure. Bienvenue et al. [2] showed that there exist two partial computable functions with the same push-forward measure but differing images of random sequences.

A related paper by Barmpalias and Shen [1] explores a notion of randomness for c.e. semimeasures different from the one proposed by Bienvenu et al. [2]. Under additional conditions, they establish a result analogous to no-randomness-from-nothing.

We clarify the situation by defining several concepts of randomness and examining the hierarchical structure among them. Our findings are best viewed in comparison with the hierarchy of randomness concepts considered in the context of partial randomness. Hudelson's doctoral thesis [3] serves as a useful survey on this topic.

Now we define some randomness notions with respect to c.e. semimeasures μ. We consider the following four notions.

(I) KA-f-complexity.
(II) Strong K-f-complexity.
(III) K-f-complexity.
(IV) f-ML-randomness.

The former three notions have been appeared in Hudelson [3, Page 13]. The notion (IV) has been defined in Bienvenue et al. [2]. The notion (I) with additional conditions are considered in Barmpalias and Shen [1].

Let $f : 2^{<\omega} \to [0, \infty]$ be a function. We intend to let $f(\sigma) = -\log \mu(\sigma)$ for a c.e. semimeasure μ. Thus, f is usually a upper semi-computable function.

Date: 2025-05-31 14:22:09 JST.
2010 *Mathematics Subject Classification.* 03D32,68Q30.
Key words and phrases. Keywords.

Definition 1 (complexity)**.** A sequence $X \in 2^\omega$ is called *KA-f-complex* if

$$K(X \upharpoonright n) > f(X \upharpoonright n) - O(1).$$

A sequence $X \in 2^\omega$ is called *strongly K-f-complex* if

$$K(X \upharpoonright n) - f(X \upharpoonright n) \to \infty \text{ as } n \to \infty.$$

A sequence $X \in 2^\omega$ is called *K-f-complex* if

$$K(X \upharpoonright n) > f(X \upharpoonright n) - O(1).$$

For a definition of ML-randomness for a c.e. semimeasure, we use a sequence of strings rather than a c.e. open set.

Definition 2 (ML-randomness)**.** A *f-ML-test* is a sequence of uniformly c.e. sets $S_n \subseteq 2^{<\omega}$ such that $\sum_{\sigma \in S_n} 2^{-f(\sigma)} \leq 2^{-n}$ for all $n \in \omega$. A sequence $X \in 2^\omega$ is called *f-ML-random* if $X \notin \bigcap_n [S_n]$ for each f-ML-test $(S_n)_n$.

Some researchers have called this notion dwt-f-randomness.

The main goal is to show the following implications:

$$\text{(I)} \Rightarrow \text{(II)} \Rightarrow \text{(III)} \Rightarrow \text{(IV)}.$$

Each implication is strict.

References

1. George Barmpalias and Alexander Shen, *The Kučera–Gács theorem revisited by Levin*, Theoretical Computer Science **947** (2023), 113693.
2. Laurent Bienvenu, Rupert Hölzl, Christopher P. Porter, and Paul Shafer, *Randomness and semimeasures*, Notre Dame Journal of Formal Logic **58** (2017), no. 3, 301–328.
3. W.M. Phillip Hudelson, *Partial randomness and kolmogorov complexity*, Phd dissertation, The Pennsylvania State University, University Park, PA, May 2013, Submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy.

(K. Miyabe) Meiji University, Japan
Email address: research@kenshi.miyabe.name