CONSTRUCTIBLE FAILURES OF ERDŐS-VOLKMANN FOR RINGS

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1. Introduction

Set-theoretical axioms and the structure of the real line are deeply intertwined. The investigation of their relationship pitches the definable structure of sets of real numbers against the behaviour of non-constructive existence axioms. The former camp is classically represented by the Borel sets; the latter by the Axiom of Choice. Let P be a property of sets of reals. The following pattern has emerged frequently:

(1)	ZFC	P holds for every Borel set
(2)	$\mathsf{ZFC} + \mathsf{CH}$	P fails for some non-Borel set
(3)	$\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}$	P holds for every set
(4)	$\mathsf{ZFC} + V{=}L$	P fails for some co-analytic set

Examples of properties following this pattern include the perfect set property and the property of Baire, Marstrand projection, and the Solecki dichotomy [2, 6, 13, 16, 12]. Normally, (3) follows from (1) by expressing P game-theoretically. Then (1) follows from Borel determinacy, and (3) follows from AD; (2) is often a diagonalisation argument. But what is the *least* complexity class for which (2) holds, assuming ZF? The complexity of CH-constructed "counterexamples" in (2) is not finely calibrated. This leads to the following research question:

"Find the least Γ such that there consistently exists $X \in \Gamma$ for which P fails."

Z. Vidnyánszky [14] has generalised a method first used by P. Erdős, K. Kunen and R. D. Mauldin [4], and by A. Miller [11] to recursively construct $\boldsymbol{\Pi}^1_1$ sets of reals, assuming $V{=}L$; this is a fruitful tool to establish item (4) in the table above. Notably, Vidnyánszky's theorem lends itself to applications in fractal geometry. Via Lutz' and Lutz' point-to-set-principle [7], algorithmic randomness and effective dimensions of reals can be used to construct sets of pathological fractal properties. hence, we aim to contribute to the general research programme of structurally classifying the proof-theoretic strength of ZF for regularity properties of sets of reals in the context of fractal geometry. Here, we focus on the Erdős-Volkmann-problem for rings.

2. Erdős-Volkmann for Rings

The Erdős-Volkmann-problem is a ring-theoretical problem originally posed by B. Volkmann [15] and partially resolved by P. Erdős and B. Volkmann [5], which contrasts the algebraic structure of subrings of $\mathbb{R}$ with their geometric structure. To state their work, let Γ denote a pointclass (e.g. Borel $= \boldsymbol{\Delta}^1_1$ or analytic $= \boldsymbol{\Sigma}^1_1$). We denote by $P_{\mathsf{ring}}(\Gamma)$ the property:

If $B \subseteq \mathbb{R}$ is a proper subring of $\mathbb{R}$ and $B \in \Gamma$, then $\dim_H(B) = 0$ or $B = \mathbb{R}$.

In 1966, Erdős and Volkmann showed that, for Borel sets, the notion of subgroup is too weak to preserve high geometric structure as characterised by $P_{\mathsf{ring}}(\boldsymbol{\Delta}^1_1)$ [5]:

Theorem 2.1 (ZF, Erdős-Volkmann-Theorem)**.** *For every $s \in [0,1]$ there exists a Borel subgroup $G \leq \mathbb{R}$ such that $\dim_H(G) = s$. In other words, $P_{\mathsf{group}}(\boldsymbol{\Delta}^1_1)$ is false.*

For sub*rings*, the situation is different, as was shown by G. Edgar and C. Miller [3] and independently Bourgain [1]:

Theorem 2.2 (ZF, Edgar-Miller-Bourgain-Theorem)**.** *If $B \subseteq \mathbb{R}$ is a proper subring of $\mathbb{R}$ that is $\boldsymbol{\Sigma}^1_1$ then $\dim_H(B) = 0$ or $B = \mathbb{R}$. In other words, $P_{\mathsf{ring}}(\boldsymbol{\Sigma}^1_1)$ is true.*

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R. O. Davies contributed item (2) by constructing subrings of arbitrary dimension. By the Edgar-Miller-Bourgain-Theorem, these subrings are not $\utilde{\mathbf{\Sigma}}^1_1$. However, Davies' proof is unpublished (cf. [9, p. 167]). In 2016, Mauldin provided the details by "completing an attack first discovered by Roy Davies" [10] and showed:

Theorem 2.3 (CH, Davies-Mauldin-Theorem)**.** *For every $s \in (0,1)$ there exists a proper subring $B \subseteq \mathbb{R}$ for which $\dim_H(B) = s$.*

This leaves items (3) and (4) open—this is one interpretation of the Erdős-Volkmann-ring-problem.

3. Our Contribution

We report on ongoing work to establish item (4) for the Erdős-Volkmann ring problem. Precisely, we aim to construct, for every $s \in (0,1)$, a proper $\utilde{\mathbf{\Pi}}^1_1$-subring $A \subseteq \mathbb{R}$ of Hausdorff dimension s. Such a result would prove that ZFC is not powerful enough to prove $P_{\mathsf{ring}}(\utilde{\mathbf{\Pi}}^1_1)$. The difficulty lies in the Davies-Mauldin-proof, which cannot be trivially effectivised to make use of Vidnyánszky's theorem alongside the point-to-set-principle. We explain these difficulties, which are both algebraic and algorithmic, and motivate possible workarounds. For instance, a classification of sets of bounded Hausdorff dimension, in the style of Marcone and Valenti [8] could be useful in transferring the Davies-Mauldin proof into Vidnyánszky's framework.

Further, it is our hope to isolate a criterion which, especially in the context of fractal geometry, gives a uniform description of those properties P for which item (4) holds. The connection between classical Hausdorff dimension and algorithmic randomness appears fruitful to provide such a characterisation. This would be of additional interest as the Erdős-Volkmann problem is also influenced by additional algebraic structure.

References

[1] J. Bourgain. On the Erdős-Volkmann and Katz-Tao ring conjectures. *Geom. Funct. Anal.*, 13(2):334–365, 2003.

[2] R. O. Davies. Two counterexamples concerning Hausdorff dimensions of projections. *Colloq. Math.*, 42:53–58, 1979.

[3] G. A. Edgar and C. Miller. Hausdorff dimension, analytic sets and transcendence. *Real Anal. Exchange*, 27(1):335–339, 2001/02.

[4] P. Erdős, K. Kunen, and R. D. Mauldin. Some additive properties of sets of real numbers. *Fund. Math.*, 113(3):187–199, 1981.

[5] P. Erdős and B. Volkmann. Additive Gruppen mit vorgegebener Hausdorffscher Dimension. *J. Reine Angew. Math.*, 221:203–208, 1966.

[6] A. S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[7] J. H. Lutz and N. Lutz. Algorithmic information, plane Kakeya sets, and conditional dimension. *ACM Trans. Comput. Theory*, 10(2):Art. 7, 22, 2018.

[8] A. Marcone and M. Valenti. On the descriptive complexity of Salem sets. *Fund. Math.*, 257(1):69–93, 2022.

[9] P. Mattila. *Geometry of sets and measures in Euclidean spaces: fractals and rectifiability*. Number 44 in Cambridge studies in advanced mathematics. Cambridge University Press, Cambridge [England] ; New York, 1995.

[10] R. D. Mauldin. Subfields of **R** with arbitrary Hausdorff dimension. *Mathematical Proceedings of the Cambridge Philosophical Society*, 161(1):157–165, July 2016.

[11] A. W. Miller. Infinite combinatorics and definability. *Ann. Pure Appl. Logic*, 41(2):179–203, 1989.

[12] L. Richter. Co-analytic counterexamples to Marstrand's projection theorem, 2023. Submitted. arXiv:2301.06684.

[13] S. Solecki. Decomposing Borel sets and functions and the structure of Baire class 1 functions. *J. Amer. Math. Soc.*, 11(3):521–550, 1998.

[14] Z. Vidnyánszky. Transfinite inductions producing coanalytic sets. *Fund. Math.*, 224(2):155–174, 2014.

[15] B. Volkmann. Eine metrische Eigenschaft reeller Zahlkörper. *Mathematische Annalen*, 141(3):237–238, 1960.

[16] J. Zapletal. Descriptive set theory and definable forcing. *Mem. Amer. Math. Soc.*, 167(793):viii+141, 2004.

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