A Verified Power-Series Method for Multivariate IVPs*

Holger Thies[1]

[1]Kyoto University, Japan

We formalize and verify an algorithm for rigorously solving initial value problems (IVPs) for systems of multivariate analytic ordinary differential equations (ODEs) based on ideas from computable analysis, within the Rocq (formerly Coq) proof assistant. The goal is to provide verified computational solutions with explicit error control based on constructive implementations of real numbers in the proof assistant. More precisely, we consider IVPs for systems of (autonomous) first-order analytic ODEs of the form

$$\dot{\vec{y}}(t) = \vec{f}(\vec{y}(t)), \quad \vec{y}(0) = \vec{y_0} \tag{1}$$

with $\vec{f} : U \subseteq \mathbb{R}^d \to \mathbb{R}^d$.

In the context of computable analysis often a method based on computing the *infinite* Taylor series expansion at a point is used [BKM16, KST18, SSTZ21] for analytic IVPs. This method is efficient in this framework as error bounds decrease exponentially in the order of the expansion and therefore (in contrast to e.g. the Euler method or Runge-Kutta methods) allows feasible computation for high precision [Mül95]. The method is based on the ODE version of the Cauchy-Kovalevskaya theorem and its classical proof (see e.g. [Tes12, Chapter 4]). The main idea is to compute the power series of the solution by recursively determining higher-order derivatives, that is, we define a sequence $F^{[n]} : \mathbb{R}^d \to \mathbb{R}^d$ of multivariate analytic functions by

$$F^{[0]}(\vec{x}) = \vec{x}, \quad F^{[n+1]}(\vec{x}) = \vec{f}(\vec{x}) \cdot \mathrm{D}\big(F^{[n]}\big)(\vec{x}),$$

where D denotes the Jacobian. The resulting infinite series $\vec{y}(t) = \sum_{i=0}^{\infty} \frac{1}{n!} \vec{F}^{[n]}(y_0) t^n$ converges effectively to the unique solution of (1). Further, rigorous error bounds can be computed explicitly from simple bounds on the right-hand side function f. Note that the resulting power series is never finite except for trivial cases.

Previous work [PT24] by Park and the author presented a formalization of this method in Rocq as part of the certified exact real computation library cAERN [KPT24]. However, the implementation only supports *one-dimensional polynomial* ODEs, and thus has limited practical applications. Furthermore, cAERN's main purpose is the extraction of efficient and certified programs for the exact real computation library AERN and it thus does not support direct computation within the proof assistant. Although extracted programs are typically more efficient than computing inside of Rocq, this feature can still be useful, especially since it allows to include the results of computations inside formal proofs.

In this talk, we present a new Rocq formalization[1] which is applicable to systems of *arbitrary* dimension and to generic analytic right-hand side functions. Furthermore, instead of using a concrete type for real numbers and functions, we use type classes and setoids to abstractly specify which kind of properties are needed. The type-classes can be instantiated with any type satisfying these properties, making the formalization flexible and easily adaptable to different formalizations of constructive real numbers in Rocq, including the implementation in the Rocq standard library. The development is modular, giving precise control over the properties required in different parts of the formalization. This makes it straightforward to

*This work was supported by JSPS KAKENHI Grant Numbers JP23K28036, and JP24K20735.

[1]`https://github.com/holgerthies/taylor_rocqs`

adapt the solver not only to exact real-number types but also to discrete numeric types such as rationals or arbitrary-precision floating-point numbers, allowing to compute exact approximations with rigorous error bounds. We further demonstrate how the results can be extended to interval arithmetic, resulting in a verified solver that efficiently computes rigorous interval enclosures for ODE trajectories.

To illustrate the practicality of our formalization, we instantiate and benchmark it using multiple concrete constructive real number frameworks, including the standard library's Cauchy reals, an implementation from the CoRN library [CFGW04], and an efficient interval version based on the CoqInterval library [MDM16], and test the implementation on several classical and practically relevant example ODEs.

References

[BKM16] Franz Brauße, Margarita Korovina, and Norbert Th Müller. Towards using exact real arithmetic for initial value problems. In *Perspectives of System Informatics: 10th International Andrei Ershov Informatics Conference, PSI 2015, in Memory of Helmut Veith, Kazan and Innopolis, Russia, August 24-27, 2015, Revised Selected Papers 10*, pages 61–74. Springer, 2016.

[CFGW04] Luís Cruz-Filipe, Herman Geuvers, and Freek Wiedijk. C-CoRN, the constructive Coq repository at Nijmegen. In *International Conference on Mathematical Knowledge Management*, pages 88–103. Springer, 2004.

[KPT24] Michal Konečný, Sewon Park, and Holger Thies. Extracting efficient exact real number computation from proofs in constructive type theory. *Journal of Logic and Computation*, page exae066, 10 2024.

[KST18] Akitoshi Kawamura, Florian Steinberg, and Holger Thies. Parameterized complexity for uniform operators on multidimensional analytic functions and ODE solving. In *International Workshop on Logic, Language, Information, and Computation*, pages 223–236. Springer, 2018.

[MDM16] Érik Martin-Dorel and Guillaume Melquiond. Proving tight bounds on univariate expressions with elementary functions in coq. *Journal of Automated Reasoning*, 57:187–217, 2016.

[Mül95] Norbert Th Müller. Constructive aspects of analytic functions. In *Proc. Workshop on Computability and Complexity in Analysis*, volume 190, pages 105–114, 1995.

[PT24] Sewon Park and Holger Thies. A coq formalization of taylor models and power series for solving ordinary differential equations. In Yves Bertot, Temur Kutsia, and Michael Norrish, editors, *15th International Conference on Interactive Theorem Proving, ITP 2024, September 9-14, 2024, Tbilisi, Georgia*, volume 309 of *LIPIcs*, pages 30:1–30:19. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2024.

[SSTZ21] Svetlana Selivanova, Florian Steinberg, Holger Thies, and Martin Ziegler. Exact real computation of solution operators for linear analytic systems of partial differential equations. In *Computer Algebra in Scientific Computing: 23rd International Workshop, CASC 2021, Sochi, Russia, September 13–17, 2021, Proceedings 23*, pages 370–390. Springer, 2021.

[Tes12] G. Teschl. *Ordinary Differential Equations and Dynamical Systems*. Graduate studies in mathematics. American Mathematical Society, 2012.