PSPACE-Completeness of the Reachability Relation of Robust Dynamical Systems

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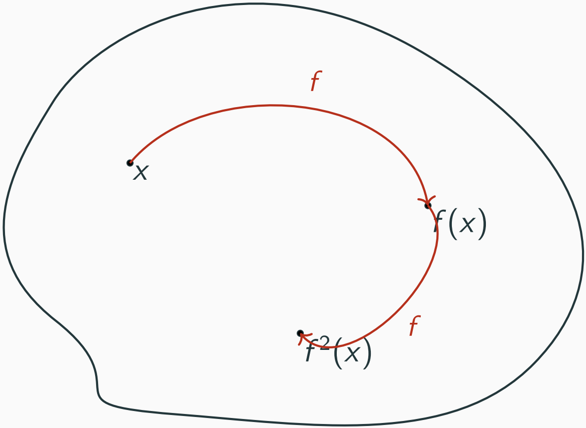
Introduction

Introduction: General Dynamical Systems



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- Reachability is undecidable...
- ... more precisely, it is not co-computably enumerable;
- It becomes decidable for a sub-class of systems: Robust Dynamical Systems.

Computability

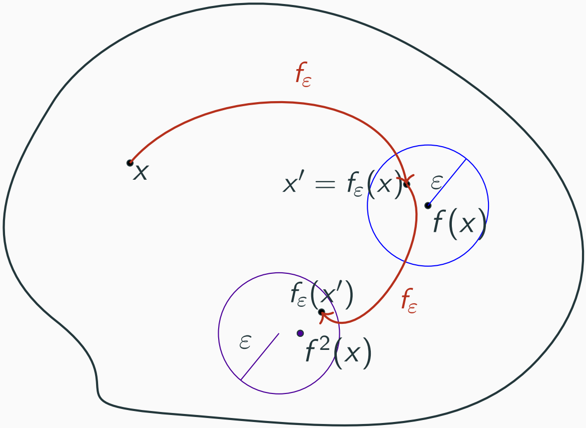
- Informal conjecture: undecidability in verification does not happen for robust systems.

- Undecidabilty is related to non-robustness of the systems (Asarin, Bouajjani)

  $\rightarrow$ with proper notion of robusteness: no sensitivity to an arbitrarly small perturbation.

Robust Dynamical Systems: Definitions and Properties

Robust Dynamical Systems



Robustness

Consider $R^{\mathcal{H}}_{\omega}(x, y) = \bigcap_{\epsilon > 0} R^{\mathcal{H}}_{\epsilon}(x, y)$

- Say $R^{\mathcal{H}}$ is robust when $R^{\mathcal{H}}_{\omega} = R^{\mathcal{H}}$:

  $R^{\mathcal{H}}$ robust $\Rightarrow$ $R^{\mathcal{H}}$ computable.

- Reachability in robust dynamical systems is *decidable* (Asarin & Bouajjani)

PSPACE **on polynomially robust dynamical systems**

Theorem (B., Bournez CSL 2024)

Take a locally Lipschitz system $\mathcal{H}$, with $f : X \to X$ polynomial-time computable, with X a closed rational box. Then, for $p : \mathbb{N} \to \mathbb{N}$ a polynomial, $R^{\mathcal{H}}_{2^{-p(n)}} \subseteq \mathbb{Q}^d \times \mathbb{Q}^d \times \mathbb{N} \in \mathsf{PSPACE}$, with n the precision of the computation.

$\rightarrow$ works also for dynamical systems over the reals

Example
Simulation of Turing machines with continuous PAMs.

Toward PSPACE-**Completeness**

Encoding [Kawamura & Cook '12]

- $f : \Sigma^* \to \Sigma^*$ regular $\Leftrightarrow$ for all $u, v \in Dom(f)$ such that $|u| \leq |v|$, then $|f(u)| \leq |f(v)|$.

- **Reg** : set of regular functions
- **Pred** : set of $\{0, 1\}$-valued regular functions

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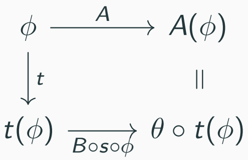
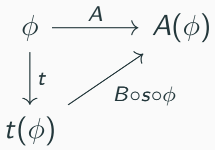
- **Reg** : set of regular functions
- **Pred** : set of $\{0, 1\}$-valued regular functions

$\to$ regular functions give a proper measure of length, especially for oracles

Definition (Kawamura-Cook)
Let A and B be decision problems, or multi-functions from **Reg** to **Pred**. We write $A \leq_m^2 B$ if there exists $s, t \in \mathsf{FPTIME}^2$ such that for any $\phi \in Dom(A)$, we have $s(\phi) \in Dom(B)$ and for $\theta = B(s(\phi))$ the function $\theta \circ t(\phi) = A(\phi)$.



or



An example of complete problem

Basic PSPACE^2:

Input: $< M, \bar{\mu}, \phi >$, with M a Type-2 TM, $\bar{\mu} \in$ **Reg** such that $\bar{\mu}(u) = 0^{\mu(|u|)}$ where $\mu : \mathbb{N} \to \mathbb{N}$ a non-decreasing polynomial bounding the space of M, $\phi \in$ **Reg** and a string u.

Output: Does M, with an oracle ϕ, accept u?

The case of Discrete-Time Dynamical Systems

A previous result on FPTIME

$$\overline{\mathbb{LDL}^\circ} = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \tanh, \frac{x}{2}, \frac{x}{3};$$

composition, *linear length ODE*, *effective limit*]

Theorem (B., Bournez, MFCS23)

$$\overline{\mathbb{LDL}^\circ} \cap \mathbb{R}^{\mathbb{R}} = \mathrm{FPTIME} \cap \mathbb{R}^{\mathbb{R}}$$

Discrete Reach Robust Lipschitz (DRRL):

Input: A discrete-time polynomially robust dynamical system $\mathcal{H} = (X, g)$, X a closed rational box, $(g : X \to X) \in \mathbb{LDL}^{\circ} + \Phi$ Lipschitz, $p \in \mathbb{N}$, $\mathbf{x}, \mathbf{y} \in X$

Output: $R^{\mathcal{H}}_{2^{-p}}(\mathbf{x}, \mathbf{y})$

with $\Phi : \gamma(w) \to \gamma(\phi(w))$, $\gamma : \Sigma^* \to \mathbb{R}$, $w \in \Sigma^*$, $\phi \in$ **Reg**

Discrete-Time

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Theorem

DRRL *is* $\mathrm{PSPACE}^2 - \leq^2_m$*-hard.*

Proof Ideas: Reduction

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Output: Does M, with an oracle ϕ, accept u?

We prove that: **Basic** PSPACE^2 $\leq_m^2$ **DRRL**

Proof Ideas: Encoding

Transforming the execution of the TM into a dynamical system:

- Domain $\to$ encoding of the space of the possible configurations
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  - t $\equiv$ encoding of the execution;
  - s $\equiv$ maps the real encoding the configuration C_i to the real encoding C_{i+1} (related to the oracle);
  - Prove it is computable in polynomial time
  - Prove the underlying function is Lipschitz

The case of Continuous-Time Dynamical Systems

Continuous Reach Robust Lipschitz (CRRL):

Input: A continuous-time polynomially robust dynamical system $\mathcal{H}_C = (X_C, g_C)$, with $X_C \subseteq \mathbb{R}$ a closed rational box, $g_C : X_C \to X_C \in \mathbb{LDL}^\circ$ Lipschitz, $p \in \mathbb{N}$, $\mathbf{x}, \mathbf{y} \in X_C$

Output: $\mathfrak{R}_{2^{-p}}^{\mathcal{H}_C}(\mathbf{x}, \mathbf{y})$

Theorem

CRRL *is* PSPACE^2-$\leq_m^2$*-hard.*

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We prove that: **DRRL** $\leq_m^2$ **CRRL**

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We prove that: **DRRL** $\leq_m^2$ **CRRL**

- We use the Branicky trick:
  - t $\equiv$ encoding of the Branicky trick in our context;
  - s $\equiv$ maps a real in the solution of the discrete ODE to the next point
  - Prove it is computable in polynomial time

Conclusion

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- **DRRL** is PSPACE^2-$\leq_m^2$-complete.
- **CRRL** is PSPACE^2-$\leq_m^2$-complete.
- Having such completeness results allows us to have a better understanding of those complexity classes.