Computable Bases

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The Presubbase Theorem

Admissible representations

Given a represented space X, we consider its neighborhood map

$$\mathcal{U} : X \to \mathcal{O}\mathcal{O}(X), x \mapsto \{U \in \mathcal{O}(X) : x \in U\}.$$

This map is always computable and continuous if $\mathcal{O}(X)$ is the final topology of X and $\mathcal{O}\mathcal{O}(X)$ the respective Scott topology.

For every represented space X there is a canonical representation of $\mathcal{O}(X)$ using the fact that $U \in \mathcal{O}(X) \iff \chi_U : X \to \mathbb{S}$ is continuous.

- X is a Kolmogorov space (T_0 space) $\iff$ $\mathcal{U}$ is injective.
- X is a continuous Kolmogorov space (admissible) $:\iff$ $\mathcal{U}$ is a topological embedding.
- X is a computable Kolmogorov space (computably admissible) $:\iff$ $\mathcal{U}$ is a computable embedding.

These definitions are due to Matthias Schröder (2002) and extensions of the original notion of admissibility by Kreitz and Weihrauch (1985).

For admissible representations continuity and continuity with respect to representation are equivalent properties.

The Theorem of Kreitz-Weihrauch

Given a countable subbase $B : \mathbb{N} \to \mathcal{O}(X)$ for a T_0 space X, by

$$\delta^B(p) = x :\iff \mathrm{range}(p) - 1 = \{n \in \mathbb{N} : x \in B_n\}$$

we can define the subbase representation of $\delta^B :\subseteq \mathbb{N}^\mathbb{N} \to X$ of X.

Theorem (Kreitz-Weihrauch 1985)

If $B : \mathbb{N} \to \mathcal{O}(X)$ is a subbase of a T_0 space X, then the subbase representation δ^B is admissible with respect to the topology of X.

This theorem has a lot of useful applications, e.g.:

- We want to prove that some representation of the reals ρ is admissible with respect to the Euclidean topology.
- Just prove that it is topologically equivalent to the subbase representation δ^B for the subbase B that lists all rational intervals (q, r) with rationals $q < r$.

Question

How can this theorem be generalized to a non-countable setting?

Presubbase representations

For general subbases $B : Y \to \mathcal{O}(X)$ we want to define

$$\delta^B(p) = x :\iff \delta_{\mathcal{O}(X)}(p) = \{y \in Y : x \in B_y\}.$$

When is this representation actually well-defined?

Definition

Let X be a set. We call a family $(B_y)_{y \in Y}$ a presubbase for X, if Y is a represented space and its *transpose*

$$B^{\mathsf{T}} : X \to \mathcal{O}(Y), x \mapsto \{y \in Y : x \in B_y\}$$

is well-defined and injective.

This is exactly what is needed to make δ^B well-defined. We obtain

$$\delta^B = (B^{\mathsf{T}})^{-1} \circ \delta_{\mathcal{O}(Y)}.$$

Note that any countable subbase $B : \mathbb{N} \to \mathcal{O}(X)$ of a T_0 topology is a presubbase!

Bases of the form $B : Y \to \mathcal{O}(X)$ with general Y have also been considered by de Brecht, Selivanov and Schröder (2016).

The Presubbase Theorem

Theorem

Let $(B_y)_{y\in Y}$ be a presubbase of a set X. Then (X,δ^B) is a computable Kolmogorov space and δ^B is admissible with respect to the topology τ on X with the base of sets X and $\bigcap_{y\in K} B_y$ for every compact $K\subseteq Y$.

Note that this theorem actually generalizes the Theorem of Kreitz-Weihrauch as for a countable subbase $B:\mathbb{N}\to\mathcal{O}(X)$ the compact sets $K\subseteq\mathbb{N}$ are exactly the finite sets.

Proof. 1. We prove that (X,δ^B) is a computable Kolmogorov space:

The definition

$$\delta^B=(B^{\mathsf{T}})^{-1}\circ\delta_{\mathcal{O}(Y)}$$

ensures that $B^{\mathsf{T}}:X\to\mathcal{O}(Y)$ is a computable embedding, which implies that $B:Y\to\mathcal{O}(X)$ is computable too.

We need to prove that also $\mathcal{U}:X\to\mathcal{O}\mathcal{O}(X)$ is a computable embedding. Given $\mathcal{U}_x=\{U\in\mathcal{O}(X):x\in U\}\in\mathcal{O}\mathcal{O}(X)$ for some $x\in X$, we can compute

$$B^{\mathsf{T}}_x=\{y\in Y:x\in B_y\}=B^{-1}(\mathcal{U}_x)\in\mathcal{O}(Y)$$

since B is computable. And since B^{T} is a computable embedding, we can compute $x\in X$ from this set.

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Proof. 2. We prove that δ^B is admissible with respect to τ:

We already know that δ^B is admissible w.r.t. its final topology $\mathcal{O}(X)$. It is known that $\mathcal{O}(Y)$ is admissibly represented with respect to the topology generated by the sets $\mathcal{F}_K = \{U \in \mathcal{O}(Y) : K \subseteq U\}$ over all compact $K \subseteq Y$. Now we note that

$$\bigcap_{y\in K} B_y = \{x \in X : K \subseteq B^{\mathsf{T}}_x\} = (B^{\mathsf{T}})^{-1}(\mathcal{F}_K)$$

and $(B^{\mathsf{T}})^{-1}(\mathcal{O}(Y)) = X$. Hence, τ is the initial topology of B^{T} and $\delta^B = (B^{\mathsf{T}})^{-1} \circ \delta_{\mathcal{O}(Y)}$ is admissible w.r.t. τ (by results of Schröder). □

The reason we speak of *presubbases* and not *subbases* is that in the general case the topology is not the one generated by finite intersections, but by compact intersections!

Computable Presubbases and Bases

Computable presubbases and prebases

Let X and Y be represented spaces. with a presubbase $B : Y \to \mathcal{O}(X)$.

- B is called a computable presubbase if $B^{\top}$ is a computable embedding.
- B is called a computable prebase if given $K \in \mathcal{K}_{-}(X)$ we can compute $A \in \mathcal{A}_{+}(X)$ with
$$\bigcap_{y \in K} B_y = \bigcup_{y \in A} B_y.$$
- B is called a computable base if it is additionally a base.
- B is called a computable Lacombe base if
$$\bigcup : \mathcal{A}_{+}(Y) \to \mathcal{O}(X), A \mapsto \bigcup_{y \in Y} B_y$$
is computably surjective.

For the computable concepts we obtain the implications:

$$\text{Lacombe base} \implies \text{base} \implies \text{prebase} \implies \text{presubbase}.$$

For $Y = \mathbb{N}$ and computable Kolmogorov spaces X every computable base is also a Lacombe base.

Question

Is every computable base also a computable Lacombe base?

Computable Kolmogorov spaces

Theorem

Let X be a represented space. Then the following are equivalent:

1. *X is a computable Kolmogorov space.*
2. *X has a computable presubbase.*
3. *X has a computable prebase.*
4. *X has a computable base.*
5. *X has a computable Lacombe base.*
6. *$\mathrm{id} : \mathcal{O}(X) \to \mathcal{O}(X)$ is a computable Lacombe base of X.*

It seems that in the general setting the notions of a computable prebase and of a computable presubbase are the most useful ones!

Corollary (Computable prebases)

If X is a represented space that has a computable prebase that generates a topology τ, then $\mathcal{O}(X) = \mathsf{seq}(\tau)$.

Presubbases for hyper and function spaces

Proposition

Let X be a represented space and Y a computable Kolmogorov space. The following are computable presubbases:

1. $\mathcal{U} : X \to \mathcal{O}\mathcal{O}(X), x \mapsto \{U : x \in U\}$.
2. $\mathcal{F} : \mathcal{K}_{-}(X) \to \mathcal{O}\mathcal{O}(X), K \mapsto \{U : K \subseteq U\}$.
3. $\Box : \mathcal{O}(X) \to \mathcal{O}\mathcal{K}_{-}(X), U \mapsto \{K : K \subseteq U\}$.
4. $\Diamond : \mathcal{O}(X) \to \mathcal{O}\mathcal{A}_{+}(X), U \mapsto \{A : A \cap U \neq \emptyset\}$.
5. $\triangleright : \mathcal{K}_{-}(X) \times \mathcal{O}(Y) \to \mathcal{O}\mathcal{C}(X, Y), (K, U) \mapsto \{f : f(K) \subseteq U\}$.

The maps $\mathcal{F}$ and $\Box$ are even computable prebases.

Proof idea. The proof is very simple. We have very natural identities such as $\mathcal{U}^{\mathsf{T}} = \mathrm{id}_{\mathcal{O}(X)}$ and $\Box^{\mathsf{T}} = \mathcal{F}$, etc. These maps are easily seen to be computable embeddings. $\Box$

Corollary

All the space $\mathcal{O}(X)$, $\mathcal{A}_{+}(X)$, $\mathcal{K}_{-}(X)$ and $\mathcal{C}(X, Y)$ are computable Kolmogorov spaces.

Presubbases for hyper and function spaces

Proposition

Let X be a represented space and Y a computable Kolmogorov space. The following are computable presubbases:

1. $\mathcal{U} : X \to \mathcal{OO}(X), x \mapsto \{U : x \in U\}$.
2. $\mathcal{F} : \mathcal{K}_-(X) \to \mathcal{OO}(X), K \mapsto \{U : K \subseteq U\}$.
3. $\Box : \mathcal{O}(X) \to \mathcal{OK}_-(X), U \mapsto \{K : K \subseteq U\}$.
4. $\Diamond : \mathcal{O}(X) \to \mathcal{OA}_+(X), U \mapsto \{A : A \cap U \neq \emptyset\}$.
5. $\triangleright : \mathcal{K}_-(X) \times \mathcal{O}(Y) \to \mathcal{OC}(X, Y), (K, U) \mapsto \{f : f(K) \subseteq U\}$.

The maps $\mathcal{F}$ and $\Box$ are even computable prebases.

space	subbase	name of topology
$\mathcal{C}(X, Y)$	$\triangleright$	compact-open topology
$\mathcal{A}_+(X)$	$\Diamond$	lower Fell topology
$\mathcal{K}_-(X)$	$\Box$	upper Vietoris topology
$\mathcal{O}(X)$	$\mathcal{F}$	compact-open topology
$\mathcal{O}(X)$	$\mathcal{U}$	point-open topology

Presubbases for hyper and function spaces

Proposition

Let X be a represented space and Y a computable Kolmogorov space. The following are computable presubbases:

1. $\mathcal{U} : X \to \mathcal{O}\mathcal{O}(X), x \mapsto \{U : x \in U\}$.
2. $\mathcal{F} : \mathcal{K}_-(X) \to \mathcal{O}\mathcal{O}(X), K \mapsto \{U : K \subseteq U\}$.
3. $\Box : \mathcal{O}(X) \to \mathcal{O}\mathcal{K}_-(X), U \mapsto \{K : K \subseteq U\}$.
4. $\Diamond : \mathcal{O}(X) \to \mathcal{O}\mathcal{A}_+(X), U \mapsto \{A : A \cap U \neq \emptyset\}$.
5. $\triangleright : \mathcal{K}_-(X) \times \mathcal{O}(Y) \to \mathcal{O}\mathcal{C}(X, Y), (K, U) \mapsto \{f : f(K) \subseteq U\}$.

The maps $\mathcal{F}$ and $\Box$ are even computable prebases.

- Note that even though $\mathcal{U}$ is a presubbase of $\mathcal{O}(X)$, the space is not admissibly represented w.r.t. the point-open topology in general.
- If we take compact intersections though, then we obtain the prebase $\bigcap_{\mathcal{K}} \mathcal{U} = \mathcal{F}$ of the compact-open topology τ.
- This means that $\mathcal{O}(X)$ is admissibly represented with respect to τ.
- Hence $\mathcal{O}\mathcal{O}(X) = \mathrm{seq}(\tau)$, which is known to be the Scott topology.
- It also follows immediately, that $\mathcal{K}_-(X)$ is admissibly represented with respect to the upper Vietoris topology.

Topologies on hyper and function spaces

Theorem (Schröder 2002)

Let X be a represented space and let Y be an admissibly represented T_0 space.

1. $\mathcal{O}(X)$ *is endowed with the sequentialization of the compact-open topology (which is the Scott topology).*
2. $\mathcal{K}_-(X)$ *is endowed with the seq. of the upper Vietoris topology.*
3. $\mathcal{A}_+(X)$ *is endowed with the sequentialization of the lower Fell topology.*
4. $\mathcal{C}(X, Y)$ *is endowed with the seq. of the compact-open topology.*

All spaces are admissibly represented with respect to the given topologies.

Proof idea.

- The proof is immediate if where we have computable prebases.
- Only in the cases of $\mathcal{A}_+(X)$ and $\mathcal{C}(X, Y)$, where we only have computable presubbases, we need to invest additional work.
- In fact, we need to look for instance at the topology τ generated by compact intersections of the presubbase $\Diamond$ of the lower Fell topology and we need to show that $\mathcal{O}\mathcal{A}_+(X) = \mathrm{seq}(\tau)$. □

Constructions on Presubbases

Constructions of overt prebases

Proposition

Let S, R be overt and $Z \subseteq X$. If $B_X : R \to \mathcal{O}(X)$ and $B_Y : S \to \mathcal{O}(Y)$ are computable presubbases (prebases), then so are:

1. $B_{X\times Y} : R \times S \to \mathcal{O}(X \times Y), (r, s) \mapsto B_X(r) \times B_Y(s)$
2. $B_{Y^{\mathbb{N}}} : S^* \to \mathcal{O}(Y^{\mathbb{N}}), (s_1, \dots, s_n) \mapsto B_Y(s_1) \times \dots \times B_Y(s_n) \times Y^{\mathbb{N}}$
3. $B_Z : R \to \mathcal{O}(Z), r \mapsto B_X(r) \cap Z$
4. $B_{X\sqcup Y} : R \sqcup S \to \mathcal{O}(X \sqcup Y), t \mapsto \begin{cases} B_X(t) & \text{if } t \in R \\ B_Y(t) & \text{if } t \in S \end{cases}$
5. $B_{X\sqcap Y} : R \times S \to \mathcal{O}(X \sqcap Y), (r, s) \mapsto B_X(r) \cap B_Y(s)$

Corollary (Computable prebases)

For computable Kolmogorov spaces X, Y these are computable prebases:

1. $B_{X\times Y} : \mathcal{O}(X) \times \mathcal{O}(Y) \to \mathcal{O}(X \times Y), (U, V) \to U \times V$ *(product)*
2. $B_{Y^{\mathbb{N}}} : \mathcal{O}(X)^* \to \mathcal{O}(Y^{\mathbb{N}}), (U_1, \dots, U_n) \mapsto U_1 \times \dots \times U_n \times Y^{\mathbb{N}}$ *(sequences)*
3. $B_Z : \mathcal{O}(X) \to \mathcal{O}(Z), U \mapsto U \cap Z$ *(subspace)*
4. $B_{X\sqcup Y} : \mathcal{O}(X) \sqcup \mathcal{O}(Y) \to \mathcal{O}(X \sqcup Y), U \mapsto U$ *(coproduct)*
5. $B_{X\sqcap Y} : \mathcal{O}(X) \times \mathcal{O}(Y) \to \mathcal{O}(X \sqcap Y), (U, V) \mapsto U \cap V$ *(meet)*

Category of computable Kolmogorov spaces

Corollary (Closure properties of computable Kolmogorov spaces)

If X and Y are computable Kolmogorov spaces, then so are

$$X \times Y, X \sqcup Y, X \sqcap Y, Y^{\mathbb{N}}$$

and every subspace $Z \subseteq X$.

Corollary (Schröder 2002)

The category of computable Kolmogorov spaces is cartesian closed.

Corollary

Let X and Y be represented T_0 spaces and let $Z \subseteq X$ be a subspace. Then:

1. $\mathcal{O}(X \times Y) = \mathrm{seq}(\mathcal{O}(X) \otimes \mathcal{O}(Y))$,
2. $\mathcal{O}(X^{\mathbb{N}}) = \mathrm{seq}(\bigotimes_{i \in \mathbb{N}} \mathcal{O}(X))$,
3. $\mathcal{O}(Z) = \mathrm{seq}(\mathcal{O}(X)|_Z)$,
4. $\mathcal{O}(X \sqcap Y) = \mathrm{seq}(\mathcal{O}(X) \wedge \mathcal{O}(Y))$.

A Galois connection

Presubbases versus representations

The results can be seen in light of an Galois connection between two maps

- $\Delta : \mathrm{PRE}_0 \to \mathrm{REP}_0, B \mapsto \delta^B$
- $\nabla : \mathrm{REP}_0 \to \mathrm{PRE}_0, \delta \mapsto B_\delta$

where X is some fixed set and

- PRE_0 is the set of presubbases $B :\subseteq \mathbb{N}^\mathbb{N} \to 2^X$ of T_0 topologies on X.
- REP_0 is the set of representations $\delta :\subseteq \mathbb{N}^\mathbb{N} \to X$ with a T_0 final topology.
- δ^B is the presubbase representation of B.
- B^δ is the representation of $\mathcal{O}(X)$ induced by δ.

For suitably defined computable reducibilities one obtains:

Theorem (Antitone Galois connection between representations and presubbases)

Fix a representable space X. Then for every $\delta \in \mathrm{REP}_0$ and $B \in \mathrm{PRE}_0$ we have

$$\delta \leq \delta^B \iff B \leq B_\delta .$$

Proof. We obtain:

$$\begin{aligned} \delta \leq \delta^B \quad &\iff \quad B^\mathrm{T} :\subseteq X \to \mathcal{O}(\mathbb{N}^\mathbb{N}) \text{ is computable with respect to } \delta \\ &\iff \quad B :\subseteq \mathbb{N}^\mathbb{N} \to \mathcal{O}(X) \text{ is computable with respect to } \delta_{\mathcal{O}(X)} \\ &\iff \quad B \leq B_\delta , \end{aligned}$$

which proves the claim. □

Presubbases versus representations

A Galois connection induces closure operators (monads):

- $\Delta \circ \nabla : \mathsf{REP}_0 \to \mathsf{REP}_0, \delta \mapsto \delta^\bullet := \delta^{B_\delta}$ and
- $\nabla \circ \Delta : \mathsf{PRE}_0 \to \mathsf{PRE}_0, B \mapsto B_\bullet := B_{\delta^B}$.

The first closure operator was studied by Schröder (2002) and used for his proofs of some of the results presented here.

We summarize some properties of these closure operators (again for fixed X):

- $\delta \mapsto \delta^\bullet$ maps every represented T_0 space (X, δ) to a computable Kolmogorov space $X^\bullet := (X, \delta^\bullet)$ with $\delta \leq \delta^\bullet$.

  The induced underlying final topologies and the representations thereof are preserved by this operation (up to computable equivalence).
- $B \mapsto B_\bullet$ maps any presubbase B of a T_0 topology on X to a computable base $B_\bullet$ of the sequentialization $\mathsf{seq}(\tau)$ of the topology τ that is generated by the compact intersections $\bigcap_{\mathcal{K}} B$. In particular, $\mathrm{range}(B) \subseteq \mathrm{range}(B_\bullet)$.

  The operation preserves the induced presubbase representations of X (up to computable equivalence).

We also obtain:

- $\mathrm{range}(\Delta)$ is exactly the set of computably admissible representations for the set X (up to computable equivalence) and
- $\mathrm{range}(\nabla)$ is exactly the set of computable bases of the qcb_0 topologies for the set X (up to computable equivalence).

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