Nondeterminism and Analog Computation: towards a characterization of **NP**

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Overview

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  - Unification of approaches via the length of the curve
  - Existence of a computable canonical representative of an analytic function

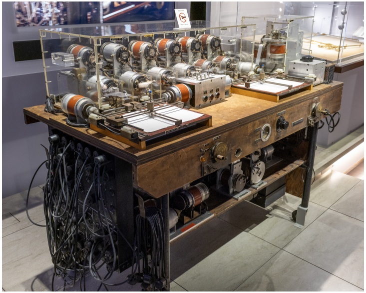


Figure 1: A differential analyser

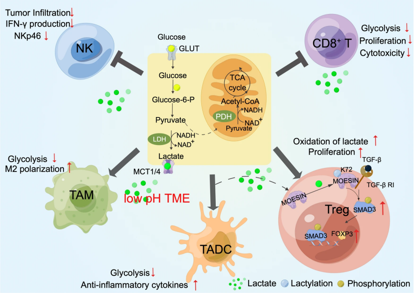


Figure 2: A natural model ([SLH22])

- 1941 : *General Purpose Analog Computer (GPAC)* ([Sha41])
- Association of basic blocks $(k, +, -, \times, \int)$ to build circuits

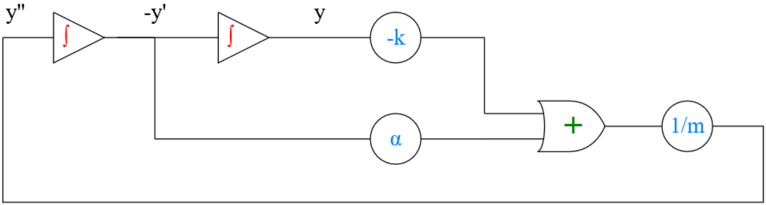
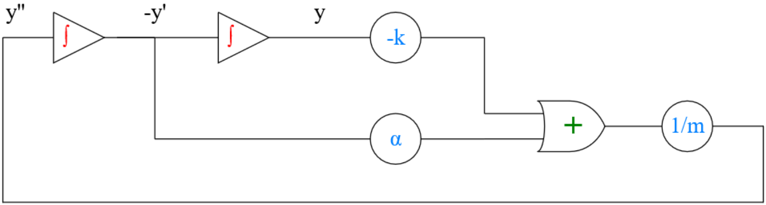


Figure 3: A damped harmonic oscillator

$$y'' = \frac{1}{m}(-ky - \alpha y')$$



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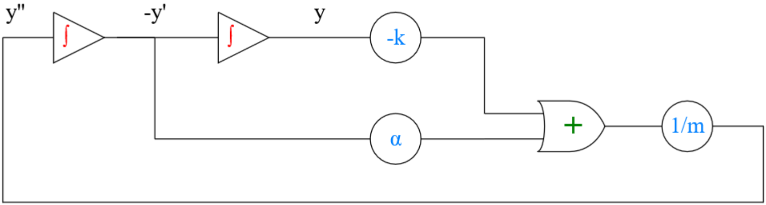

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$$y'' = \frac{1}{m}(-ky - \alpha y')$$

- Function **generated** by a GPAC : $y' = p(y), \quad p \in \mathbb{R}[X]$
- It is rather the *trace* of the computation...

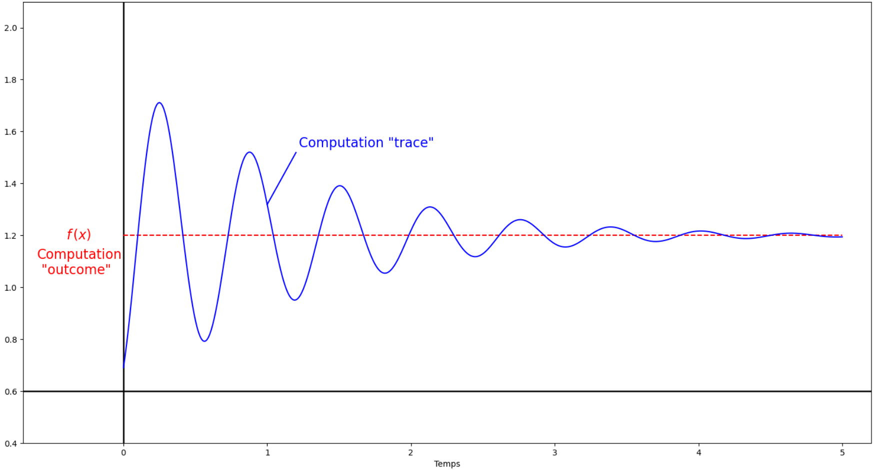


Figure 4: Computation outcome v. Computation trace

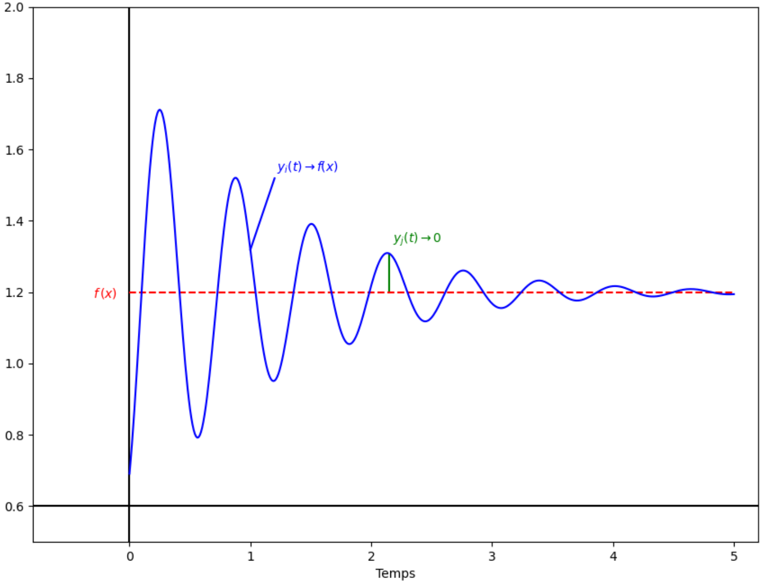
Definition

$f:[a,b]\to\mathbb{R}$ is **GPAC-computable** iff there exists $p:\mathbb{R}^{n+1}\to\mathbb{R}^n$ a polynomial, $p_0\in\mathbb{R}[X]$, $\alpha_1,\ldots,\alpha_{n-1}\in\mathbb{R}_c$ such that for $x\in[a,b]$, if $y\in\mathbb{R}^n$ is the solution of

$$Y'=p(Y,t)$$

with initial condition $Y_0=(\alpha_1,\ldots,\alpha_{n-1},p_0(x))$, we have :

$$\exists\,(i,j)\text{ such that }\begin{cases}\lim\limits_{t\to\infty} y_j(t)=0\\ |f(x)-y_i(t)|\leq y_j(t)\end{cases}$$


2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0
1
2
3
4
5
Temps
$y_i(t) \to f(x)$
$y_j(t) \to 0$
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$\square$

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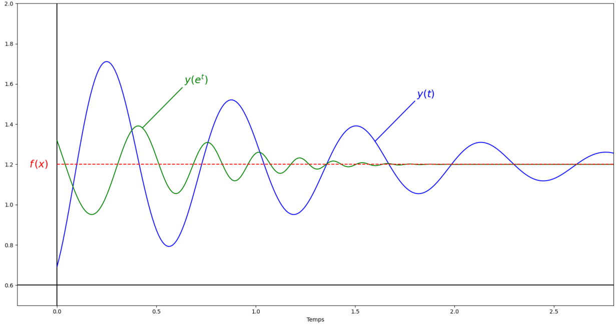

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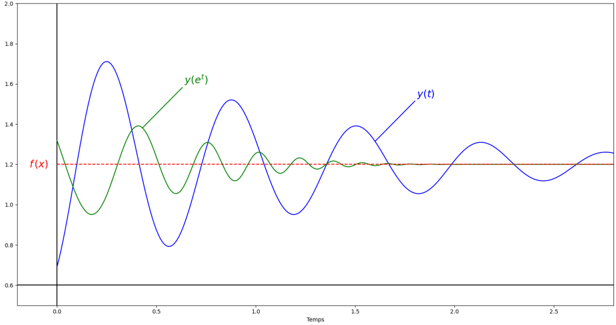

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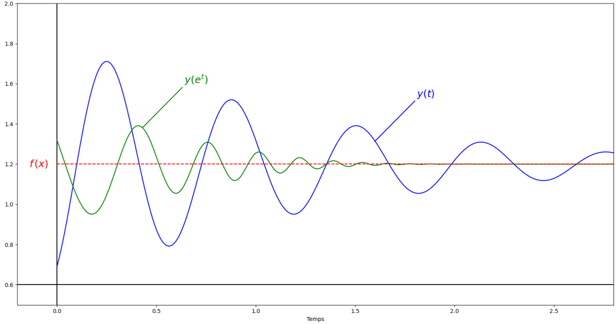

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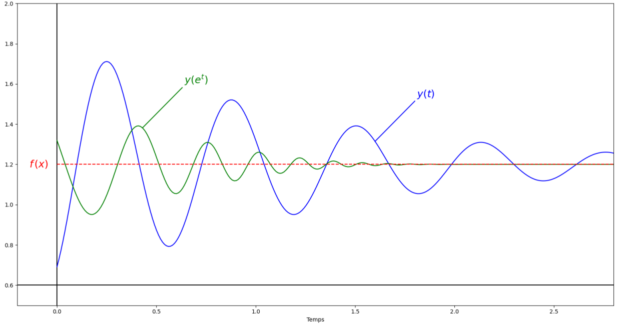

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Theorem ([Pou15])

f is (classically) polynomial-time computable

$$\Longleftrightarrow$$

f is GPAC-computable by a GPAC of polynomial length

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In general, evaluation of a power series is not computable from its coefficients.

- We need more information

$f : [0,1] \to \mathbb{R}$ analytic.

Definition

A name for f consists of $(A, K, (a_{m,n})_{m \leq 2K, n \geq 0})$ such that :

- $a_{m,n}$ is the n-th Taylor's coefficient of f around $\frac{m}{2K}$
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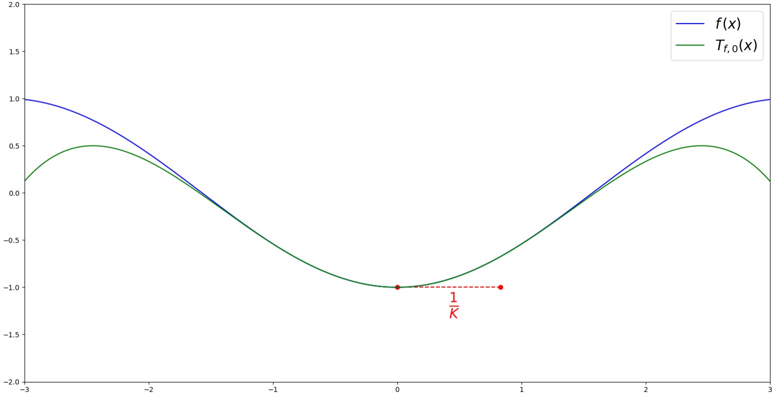

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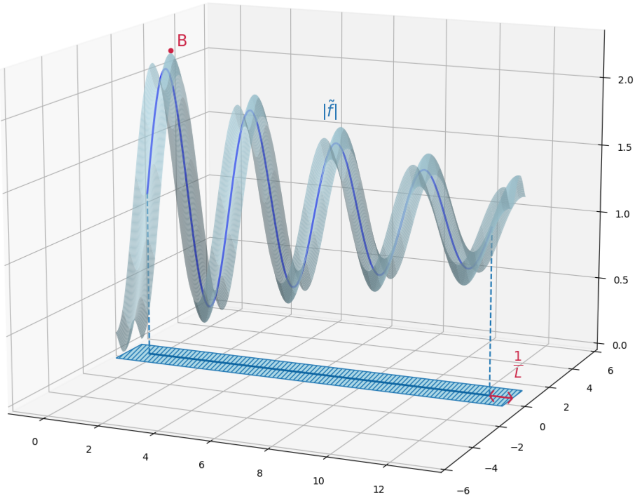
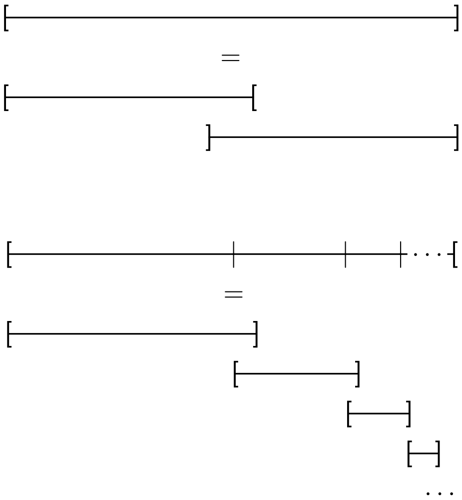


- More natural to talk about analytic functions on open intervals



Theorem ([Thi18])

$F : [0,1] \to \mathbb{R}$ analytic, A, K the parameters of its name.
The solution of $y' = F(y)$ (with initial condition $y_0 \in [0,1]$) can be approximated up to 2^{-n} in a time polynomial in $(n + A + K)$.

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- Extension to **NP** ?

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Lemma (Definition of NP by certificate)

$$L \in \mathbf{NP} \iff \exists L' \in \mathbf{P} \text{ such that } \forall x,$$
$$x \in L \iff \exists w, |w| \leq poly(|x|) \text{ and } (x, w) \in L'$$

Modification of Branicky's trick :

$$\begin{cases} z_1' = (f(z_2, A(t)) - z_1)^3 \phi(t) \\ z_2' = (z_1 - z_2)^3 \phi(\text{-}t) \end{cases}$$

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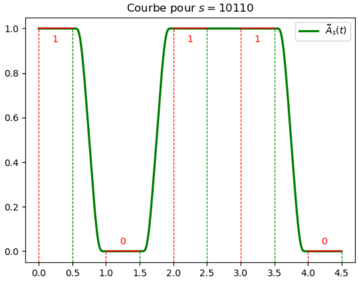

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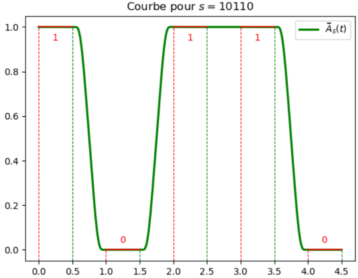

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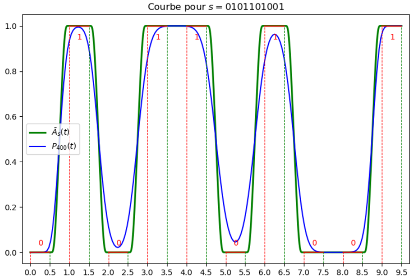

Figure 6: Analytic $A(t)$

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Theorem

*One can explicit a polynomial whose degree and coefficients are **quadratic in T** that approximates $A(t)$ at a given precision on $[0, T]$.*

Proof.

Use Bernstein polynomials. □

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- The existence of the certificate then means the existence of a polynomial in a certain class of polynomials.

Link between the two characterizations of **P** ([Pou15]/[Thi18]) ?

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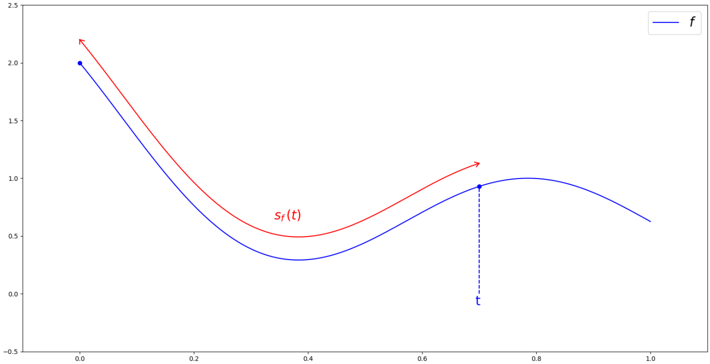
Definition (*Abscisse curviligne*)

$f \in \mathcal{C}^1([0,1],\mathbb{R})$, $L = \operatorname{len}(f)$. We call *abscisse curviligne* of f:

$$\begin{array}{rccc} s_f : & [0,1] & \longrightarrow & [0,L] \\ & x & \longmapsto & \operatorname{len}_{[0,x]}(f) \end{array}$$

That is: $s_f(t) = \int_0^t \sqrt{1+f'(x)^2}\,dx$



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- Consequence : one can compute $\mathrm{len}(f) = s_f(1)$ from a name of f, and its value is polynomial in the parameters of f.

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$$\frac{1}{A} f\left(\frac{x}{K}\right) \quad \text{(defined on } [aK, bK])$$

  - Computable, admits for parameters $(A', K') = (1, 1)$ ✓

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- Second attempt:

$$\frac{1}{A}(f \circ s_f^{-1})\left(\frac{x}{4K}\right) \quad \text{(defined on } [0, \mathrm{len}(f)K])$$

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