Speedability of computably approximable reals and their approximations

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computable approximations and computably approximable reals

- A real α is *computably approximable* if there is a computable sequence $\{a_s\}_{s\in\omega}$ of rational numbers that converges to α. Such a sequence $\{a_s\}_{s\in\omega}$ is called a *computable approximation* (of α).
- If further, $\{a_s\}_{s\in\omega}$ is nondecreasing (resp., nonincreasing), then it is called a *left-c.e.* (resp., *right-c.e.*) approximation and the real α is called a *left-c.e.* (resp., *right-c.e.*) real.
- If $\sum_s |a_{s+1} - a_s|$ is finite, then $\{a_s\}_{s\in\omega}$ is called a *d.c.e.* approximation and the real α is called a *d.c.e.* real.

We study the speedability of all this different classes of reals according to the speedability of their computable approximations.

Before we give the definition of speedability, we first give some motivation.

Solovay reducibility

For left-c.e. reals α and β, α is *Solovay reducible* to β, denoted $\alpha \leq_S \beta$, if

$(*)$ there exist a left-c.e. approximation $\{a_s\}_{s\in\omega}$ of α, a left-c.e. approximation $\{b_s\}_{s\in\omega}$ of β, and a constant c such that $\alpha - a_s \leq c(\beta - b_s)$ for all s.

Note that if α is not rational, the condition $(*)$ is equivalent to each of the following two conditions.

$(*')$ For any left-c.e. approximation $\{a'_s\}_{s\in\omega}$ of α, there exists a left-c.e. approximation $\{b'_s\}_{s\in\omega}$ of β and a constant c such that $\alpha - a'_s \leq c(\beta - b'_s)$ for all s.

$(*'')$ For any left-c.e. approximation $\{b''_s\}_{s\in\omega}$ of β, there exists a left-c.e. approximation $\{a''_s\}_{s\in\omega}$ of α and a constant c such that $\alpha - a''_s \leq c(\beta - b''_s)$ for all s.

These equivalences indicate that the notion of Solovay reducibility is robust and that, intuitively speaking,

$\alpha \leq_S \beta$ amounts to α being approximable by left-c.e. approximations at least as fast as β.

convergence rate of random left-c.e. reals

The degree structure induced by Solovay reducibility on the class of left-c.e. reals has been well studied. One important result is the following theorem by Solovay, and Kučera and Slaman.

Theorem 1 (Solovay; Kučera & Slaman)

A left-c.e. real is Solovay complete if and only if it is Martin-Löf random.

Here, a left-c.e. real is *Solovay complete* if every left-c.e. real is Solovay reducible to it.

This theorem implies that among left-c.e. reals exactly the random ones converge as slowly as possible. This phenomenon was further explored by [Barmpalias and Lewis-Pye, 2017] and [Miller, 2017].

Fix a random left-c.e. real Ω with a left-c.e. approximation $\{\Omega_s\}_{s\in\omega}$.

Definition 2

Given a d.c.e. real α with a d.c.e. approximation $\{\alpha_s\}_{s\in\omega}$, let

$$\partial\alpha = \partial\{\alpha_s\} = \lim_{s\to\infty} \frac{\alpha - \alpha_s}{\Omega - \Omega_s}.$$

Accordingly, $\partial\alpha$ is well-defined and its value is independent of the choice of $\{\alpha_s\}_{s\in\omega}$.

convergence rate of random d.c.e. reals

Theorem 3 (Miller)

$\partial\alpha = 0$ if and only if α is not random.

Corollary 4

All the d.c.e. approximations to a random d.c.e. real converge at exactly the same speed.

Proof.

Let γ be a random d.c.e. real, $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$ two d.c.e. approximations of γ. Then $\partial\{a_s\} = \partial\{b_s\} = \partial\gamma \neq 0$, i.e. ,

$$\lim_{s\to\infty} \frac{\gamma - a_s}{\Omega - \Omega_s} = \lim_{s\to\infty} \frac{\gamma - b_s}{\Omega - \Omega_s} \neq 0.$$

Then

$$\lim_{s\to\infty} \frac{\gamma - b_s}{\gamma - a_s} = \lim_{s\to\infty} \frac{\frac{\gamma - b_s}{\Omega - \Omega_s}}{\frac{\gamma - a_s}{\Omega - \Omega_s}} = 1.$$

□

definition of speedability

Instead of comparing the speed of convergence of the approximations of two different left-c.e. reals, one can also consider

- the speed of convergence of different approximations of a single left-c.e. real in general,
- the possibility of speeding up the convergence rate of a single computable approximation.

Definition 5 (speedability for computable approximations)

Let $\{a_s\}_{s\in\omega}$ be a computable approximation of a real α and $\rho \in [0,1)$ be some rational.

- If $a_s = \alpha$ for all but finitely many s, we simply say $\{a_s\}_{s\in\omega}$ is *ρ-speedable*.
- Otherwise, we say $\{a_s\}_{s\in\omega}$ is *ρ-speedable* if there is a nondecreasing computable function $f\colon \mathbb{N} \to \mathbb{N}$ such that

$$\liminf_{s\to\infty} \left| \frac{\alpha - a_{f(s)}}{\alpha - a_s} \right| \leq \rho.$$

We say $\{a_s\}_{s\in\omega}$ is *speedable* if it is ρ-speedable for some $\rho \in (0,1)$.

As long as ρ is nonzero, the actual choice of ρ in the definition does not matter by a result of [Merkle & Titov, 2020].

definition of speedability

Definition 6

Let α be a real number. Let X stand for one of the terms left-c.e., right-c.e., d.c.e., or c.a.

- α is X *speedable* if it has a speedable X approximation.
- α is *fully* X *speedable* if it is X and all of its X approximations are speedable.
- α is *strongly* X *speedable* if it has an X approximation that is speedable via the function $f(s) = s + 1$.
- α is *weakly* X *speedable* if there exist a rational $\rho \in (0, 1)$ and two X approximations $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$ of α such that

$$\liminf_{s\to\infty}\left|\frac{\alpha - b_s}{\alpha - a_s}\right| \leq \rho.$$

- α is X *0-speedable* if it has a X speedable approximation which is 0-speedable.

properties of speedability

For all $\mathsf{X} \in \{\text{left-c.e.},\ \text{right-c.e.},\ \text{d.c.e.},\ \text{c.a.}\}$, the following implications hold.

$$\text{strong } \mathsf{X} \text{ speedability} \Rightarrow \mathsf{X} \text{ speedability} \Rightarrow \text{weak } \mathsf{X} \text{ speedability}$$

$$\text{full } \mathsf{X} \text{ speedability} \Rightarrow \mathsf{X} \text{ speedability}$$

$$\mathsf{X} \text{ 0-speedability} \Rightarrow \mathsf{X} \text{ speedability}$$

Questions:

- Can any of the above implications be reversed for some X?
- What's the relation between speedability and non-randomness?

left-c.e. speedability

[Merkle & Titov, 2020] studied left-c.e. speedability and proved that for left-c.e. in place of X, the most of the implications can all be reversed.

Theorem 7 (Merkle & Titov)

For a left-c.e. real γ, the following are equivalent.

1. *γ is left-c.e. speedable.*
2. *γ is fully left-c.e. speedable.*
3. *γ is strongly left-c.e. speedable.*
4. *γ is weakly left-c.e. speedable.*

By Theorem 3, all left-c.e. random reals are not left-c.e. speedable.

Questions:

Q1: Are all nonrandom left-c.e. reals left-c.e. speedable?

Q2: Are all left-c.e. speedable reals left-c.e. 0-speedable?

- Q1 is answered in the negative by [Hölzel & Janicki, 2023].
- Q2 is answered in the negative by [Hölzel, Janicki, Merkle & Stephan, 2024].

d.c.e. speedability

For d.c.e. in place of X, we show that most of the simple implications can be reversed. Moreover, we will prove that a d.c.e. real is d.c.e. speedable if and only if it is nonrandom.

Theorem 8 (Barmpalias, Merkle, Titov, F.)

For a d.c.e. real γ, the following are equivalent.

1. *γ is d.c.e. 0-speedable.*
2. *γ is strongly d.c.e. speedable.*
3. *γ is d.c.e. speedable.*
4. *γ is weakly d.c.e. speedable.*
5. *γ is nonrandom.*

Lemma 9

If γ is a nonrandom d.c.e. real, then γ is strongly d.c.e. 0-speedable.

full d.c.e. speedability

- Due to the negative answer of Q1 by [Hölzel & Janicki, 2023], there is a nonrandom left-c.e. real which is not left-c.e. speedable.
- For this real, it is d.c.e. speedable but not fully d.c.e. speedable, since it has a left-c.e. approximation which is not speedable and all left-c.e. approximations are also d.c.e. approximations.
- Thus, d.c.e. speedability does not imply full d.c.e. speedability.

However, the implication holds when restricting attention to d.c.e. reals that are neither left-c.e. nor right-c.e. .

Theorem 10 (Barmpalias, Merkle, Titov, F.)

If a d.c.e. real is neither left-c.e. nor right-c.e., then it is fully d.c.e. speedable.

Lemma 11

Every two-sided computable approximation is speedable.

c.a. speedability

Corollary 12

If a c.a. real is neither left-c.e. nor right-c.e., then it is fully c.a. speedable.

Proposition 13

Every c.a. real has a two-sided computable approximation.

Corollary 14

Every c.a. real is c.a. speedable.

With a more involved argument, we can even show the following stronger result.

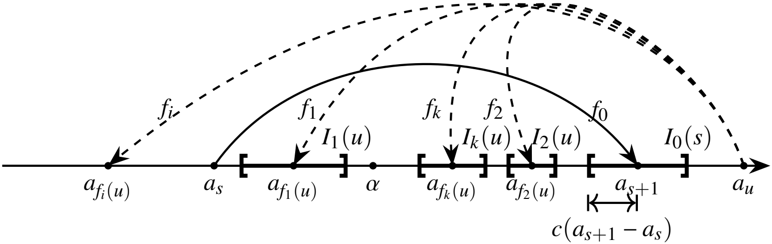
Theorem 15

Every c.a. real is strongly c.a. 0-speedable.

Proof idea of Lemma 11 (I)

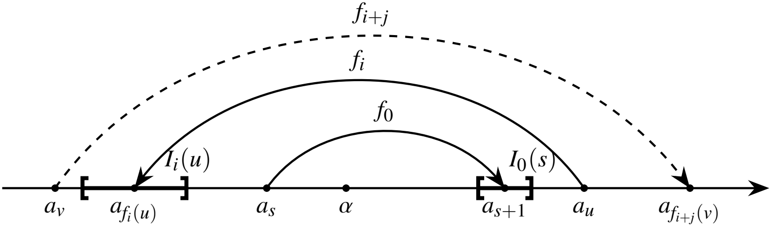
We prove by contradiction. Suppose $\{a_s\}_{s\in\omega}$ is a two-sided computable approximable to real α, and $\{a_s\}_{s\in\omega}$ is not ρ-speedable. Let $s \geq n_{2k}$ be an index such that $a_s < \alpha < a_{s+1}$.

- We inductively define $2k+1$ many computable functions $f_i \colon \mathbb{N} \to \mathbb{N}$ for $0 \leq i \leq 2k$.
- We first find the index $u \geq s+1$ such that a_u is the last one to the right of or equal to a_{s+1}.
- Then all of $a_{f_1(u)}, a_{f_2(u)}, \ldots, a_{f_k(u)}$ are to the left of a_{s+1}.
- Moreover, they must jump over the interval $[a_{s+1} - c(a_{s+1} - a_s), a_{s+1}]$. Then their distances with a_u is bounded below.
- Thus, by our arrangement of the parameters, we argue that at least one of them must be to the left of a_s.



Proof idea of Lemma 11 (II)

- Fixing one such point $a_{f_i(u)}$, we find the index $v \geq f_i(u)$ such that a_v is the last one to the left of or equal to $a_{f_i(u)}$.
- In a symmetrical way as before, we find one point $a_{f_{i+j}(v)}$ to be right of a_u.
- However this contradicts to the fact that a_u is the last one to the right of or equal to a_{s+1}.



Thank you for your attention!