Subrecursive degrees of difference representations of irrational numbers

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Our main question of interest: *can we transform one representation into another without using unbounded search*?

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The equivalence classes of representations are called S-degrees.

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- for $b \geq 2$, the *base-b expansion* of α is the function $E_b : \mathbb{N} \to \{0,1,\ldots,b-1\}$, such that

$$\alpha \;=\; \sum_{n=0}^{\infty} E_b(n) \cdot b^{-n}.$$

$E_b \preccurlyeq_S D$

Assume we have computed $E_b(1), E_b(2), \ldots, E_b(n)$ and let

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To compute $E_b(n+1)$: we search for the unique $\mathtt{D} \in \{0, 1, \ldots, b-1\}$, such that

$$D(q_n + \mathtt{D} \cdot b^{-n-1}) = 0 \quad \& \quad D(q_n + (\mathtt{D}+1) \cdot b^{-n-1}) = 1$$

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No unbounded search is used in this algorithm!

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Therefore, we have $E_b \prec_S D$.

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In other cases, as we shall see, α has a more transparent definition, justified using a growth argument.

Base-b sum approximations from below

Let us fix some base b. Any irrational number $\alpha \in (0,1)$ can be written in the form

$$\alpha = 0 + \frac{d_1}{b^{k_1}} + \frac{d_2}{b^{k_2}} + \frac{d_3}{b^{k_3}} + \dots,$$

where k_n is a strictly increasing sequence of positive integers and d_n are non-zero base-b digits, $d_n \in \{1, \dots, b-1\}$.

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The function $\hat{A}_b$, defined by $\hat{A}_b(n) = d_n b^{-k_n}$ for $n > 0$ and $\hat{A}_b(0) = 0$ is called the *base-b sum approximation from below* of the number α.

Base-b sum approximations from above

Moreover, we can write

$$\alpha = 1 - \frac{d_1'}{b^{m_1}} - \frac{d_2'}{b^{m_2}} - \frac{d_3'}{b^{m_3}} - \dots,$$

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Moreover, we can write

$$\alpha = 1 - \frac{d'_1}{b^{m_1}} - \frac{d'_2}{b^{m_2}} - \frac{d'_3}{b^{m_3}} - \dots,$$

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The function $\check{A}_b$, defined by $\check{A}_b(n) = d'_n b^{-m_n}$ for $n > 0$ and $\check{A}_b(0) = 1$ is called the *base-b sum approximation from above* of the number α.

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Results on sum approximations

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For any base b, the representations $\hat{A}_b, \check{A}_b$ and D are pairwise incomparable with respect to $\preccurlyeq_S$.

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The function $\hat{G} : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$, such that

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In [5] we constructed, given an arbitrary time bound s,
an irrational α such that $\hat{A}_b$ is primitive recursive for any fixed b,
but $\hat{G}$ is not computable in time $O(s)$.

Trace functions

A trace function from below for α is a function $T^{\uparrow} : \mathbb{Q} \cap [0,1] \to \mathbb{Q} \cap [0,1]$, such that:

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$$q > \alpha \;\; \Rightarrow \;\; q > T^{\downarrow}(q) > \alpha.$$

By symmetry, $\check{G}$ is S-equivalent to D and $T^{\downarrow}$.

Difference representations with respect to rational numbers

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For any $\alpha \in (0,1) \setminus \mathbb{Q}$ we define $\mathcal{D}\mathrm{iff}_R^{\alpha} : \mathbb{Q} \to A$ with

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Is $\mathcal{D}\mathrm{iff}_R^\alpha$ a representation of irrational numbers?

Two natural assumptions for R

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  We can compute R^α with oracle $\mathcal{D}\mathrm{iff}_R^\alpha$ in the following way:
  1. Let $a_0, \ldots, a_k = a$ be the first $k+1$ elements of A and assume we have computed $R^\alpha(a_i)$ for $i < k$.
  2. For each $b_k \in B$:
     Compute $q = P(\overline{a}, \overline{b})$, where $b_i = R^\alpha(a_i)$ for $i < k$.
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  If B is finite and P is primitive recursive, $R \preccurlyeq_S \mathcal{D}\mathrm{iff}_R$.

Difference with respect to base-b expansions

As a first example we consider $\mathcal{D}\mathrm{iff}_b$, which corresponds to the base-b expansion E_b. The base-b expansions of numbers of the form $\frac{m}{b^n}$ are assumed to end in zeros.

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It is also easy to produce a trace function from below using $\mathcal{D}\mathrm{iff}_b$: $T^{\uparrow}(q) = \sum_{i=0}^{n} E_b(i) \cdot b^{-i}$, where $n = \mathcal{D}\mathrm{iff}_b(q)$.

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In the paper [4] we prove that: $\hat{G} \prec_S \mathcal{D}\mathrm{iff}_b$ and also that $\check{G}$ and $\mathcal{D}\mathrm{iff}_b$ are subrecursively incomparable.

Difference with respect to base-b sum approximations

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We have $E_b \preccurlyeq_S \mathcal{D}\mathrm{iff}_{b\uparrow}$ with the following algorithm:
Given n and the digits $E_b(1), \ldots, E_b(n-1)$,

1. For each $\mathtt{D} \in \{1, \ldots, b-1\}$:
    1.1. Let $q = E_b(1) \cdot b^{-1} + \ldots + E_b(n-1) \cdot b^{-n+1} + x \cdot b^{-n}$.
    1.2. Let k be the number of non-zero base-b digits of q.
    1.3. If $\mathcal{D}\mathrm{iff}_{b\uparrow}(q) > k$ then output $E_b(n) = \mathtt{D}$.
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Now we also have $\mathcal{D} \preccurlyeq_S \mathcal{D}\mathrm{iff}_{b\uparrow}$. Given $q \in \mathbb{Q}$, we can compare q and α using the base-b expansions: they differ in at least one position $\leq \ell$, where ℓ is the weight of $\hat{A}_b^q(\mathcal{D}\mathrm{iff}_{b\uparrow}(q))$.

Incomparability of $\mathcal{D}\mathrm{iff}_{b\uparrow}$ with trace functions

Theorem

$$\mathcal{D}\mathrm{iff}_{b\uparrow} \not\preceq_S \hat{G}, \;\; \mathcal{D}\mathrm{iff}_{b\uparrow} \not\preceq_S \check{G}$$

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Take the following irrational number:

$$\alpha = 0.000\ldots.000010101\ldots.D_0 0010101\ldots.D_1 00101\ldots..$$

$$\uparrow d_0 \qquad\qquad \uparrow d_1 \qquad\qquad \uparrow d_2$$

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Theorem

$$\hat{A}_b \not\preccurlyeq_S \mathcal{D}\mathrm{iff}_{b\uparrow}, \quad \check{A}_b \not\preccurlyeq_S \mathcal{D}\mathrm{iff}_{b\uparrow}$$

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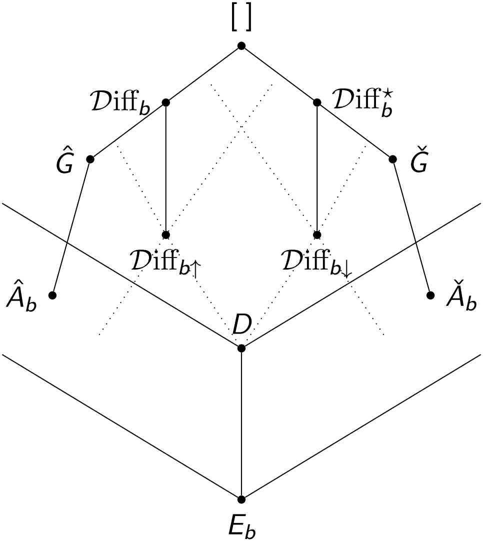
Picture of S-degrees



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Thanks for your attention!