Constant-depth Approximation of Nowhere-Differentiable Functions

Bruce Kapron

Computer Science Department, University of Victoria

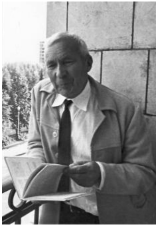
Twenty-Second International Conference on Computability and Complexity in Analysis September 24-26, 2025, Kyoto, Japan

Boolean Approximation Theory

Proposed by Kolmogorov (ICM 1962)

Approximation by Boolean circuits rather than polynomials

Focus on circuit size rather than degree



A.N. Kolmogorov 1903-1987

Boolean Approximation Theory

Work over $[-1, 1]$

Interpret $x = b_0 b_1 \ldots b_k \in \{0, 1\}^{k+1}$ as a signed dyadic rational $\overline{x} = -1 + b_0.b_1 \ldots b_k$

$C_n : \{0, 1\}^{n+1} \to \{0, 1\}^{m(n)+1}$ is an $\epsilon(n)$-approximation for $f : [-1, 1] \to [-1, 1]$ if for all $x \in [-1, 1]$ and $z \in \{0, 1\}^{n+1}$,

$$|x - \overline{z}| \leq 2^{-n} \Rightarrow |f(x) - \overline{y}| \leq \epsilon(n)$$

where $y = C_n(z)$.

Successes and Failures

Tight and nearly tight bounds for approximation of smooth and analytic functions

By entropy arguments, 2^{-n} approximation of smooth functions may require exponential size circuits (matching the upper bound)

- Constructive argument given by [Asarin 1984]
- Shannon's (nonconstructive) l.b. for Boolean functions imply Kolmogorov's l.b. [K, Ferée]

Certain analytic functions can be 2^{-n} approximated in size $\tilde{O}(n^2)$ (nearly matching lower bound)

Inverse theorem for analytic functions fails

An Inverse Theorem for Polynomial Approximation

[Bernstein 1912] gives two results for polynomial approximation with exponential convergence:

- f is analytic on [-1,1],
- f can be continued analytically to the open Bernstein ellipse E_ρ,
- f is bounded on E_ρ,

$\Rightarrow f$ is $O(\rho^{-n})$-approximated by a sequence $\{p_n\}$ where $p_n \in P_n$

Inverse theorem: f is $O(\rho^{-n})$-approximated by $\{q_n\}$ where $q_n \in P_n \Rightarrow f$ can be continued analytically to E_ρ

S.N. Bernstein 1880-1968

Failure for Circuit Size

Kolmogorov: variant of "van der Waerden's function" can be 2^{-n} approximated by $O(n^2)$-size circuits

Can we regain an inverse theorem based on other measures of circuit complexity?

[Reif, Tate 1992], [Maciel, Therien 1999] show that certain analytic functions may be approximated by constant-depth threshold circuits

Inverse theorem for circuit depth?

AC^0 and TC^0

AC^0 – constant depth circuits with $\wedge$, $\vee$, and $\neg$ gates

TC^0 – add threshold gates

$$Th_{\theta,a_1,\ldots,a_n}(x_1,\ldots,x_n) = \begin{cases} 1 & \text{if } a_1x_2 + \cdots + a_nx_n \geq \theta, \\ 0 & \text{otherwise} \end{cases}$$

where $\theta, a_1 \ldots a_n \in \mathbb{Z}$

Takagi's Continuous Nowhere-Differentiable Function

T. Takagi, A simple example of the continuous function without derivative, *Proceedings of the Physico-Mathematical Society of Japan*, ser II, Vol 1. 1903

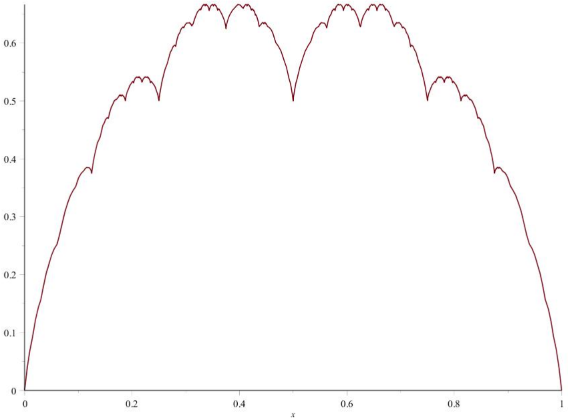
Takagi's function $\tau : [0,1] \to [0,1]$ may be defined as:

$$\tau(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} \langle\langle 2^k x \rangle\rangle.$$

where $\langle\langle y \rangle\rangle = \mathrm{dist}(y, \mathbb{N})$.

T. Takagi 1875-1960

Takagi's Function



Takagi ("Blancmange") Curve

Takagi's Function

Blancmange

Approximation in TC^0

Suppose $x, \tilde{x} \in [0,1]$ where $\tilde{x} = \sum_{i=1}^{2n+2} b_i 2^{-i}$ and $|x - \tilde{x}| \leq 2^{-(2n+2)}$.

A simple calculation gives $|\tau(x) - \tau(\tilde{x})| \leq 2^{-n}$.

But ... $\tau(\tilde{x})$ is a $(2n+2)$-term sum of $(2n+2)$-bit numbers

Iterated addition $\in TC^0$ (see, e.g., [Vollmer 1999])

τ can be $2^{-O(n)}$ approximated in TC^0

What about AC^0?

Failure of AC^0-Approximation

If $x = 0.b_1 b_2 b_3 \ldots$ and $i \geq 0$,

$$\langle\langle 2^i x \rangle\rangle = \begin{cases} 0.b_{i+1} b_{i+2} \ldots & \text{if } b_{i+1} = 0 \\ 0.\overline{b}_{i+1} \overline{b}_{i+2} \ldots & \text{if } b_{i+1} = 1 \end{cases}$$
$$= 0.(b_{i+1} \oplus b_{i+1})(b_{i+1} \oplus b_{i+2}) \ldots$$

and so

$$\frac{1}{2^i} \langle\langle 2^i x \rangle\rangle = 0.\underbrace{00 \ldots 0}_{i}(b_{i+1} \oplus b_{i+1})(b_{i+1} \oplus b_{i+2}) \ldots$$

Failure of AC^0-Approximation

When $\tilde{x} = 0.b_1 b_2 \ldots b_n 1$, for $0 \leq k < n$,

$$\langle\langle 2^k \tilde{x} \rangle\rangle = \begin{cases} 0.b_{k+1} \ldots b_{n-1} b_n 100 \ldots & \text{if } b_{k+1} = 0 \\ 0.\overline{b}_{k+1} \ldots \overline{b}_{n-1} \overline{b}_n 011 \cdots = 0.\overline{b}_{k+1} \ldots \overline{b}_{n-1} \overline{b}_n 100 \ldots & \text{if } b_{k+1} = 1 \end{cases}$$

So that

$$\begin{array}{rlllll} \tau(\tilde{x}) = 0.(b_1 \oplus b_1) & (b_1 \oplus b_2) & \ldots & (b_1 \oplus b_n) & 1 \\ + \; 0. \quad 0 & (b_2 \oplus b_2) & \ldots & (b_2 \oplus b_n) & 1 \\ + \; 0. \quad 0 & 0 & \ldots & (b_3 \oplus b_n) & 1 \\ & & \ldots & & \\ + \; 0. \quad 0 & 0 & \ldots & (b_n \oplus b_n) & 1 \\ + \; 0. \quad 0 & 0 & \ldots & 0 & 1 \end{array}$$

Failure of AC^0-Approximation

Letting c denote the nth bit of $\tau(\tilde{x})$, we have

$$b_1 \oplus \cdots \oplus b_n = \begin{cases} c \oplus b_n \oplus 1 & \text{if } n+1 \text{ is even and } \left\lfloor \frac{n+1}{2} \right\rfloor \text{ is odd} \\ c \oplus b_n & \text{if } n+1 \text{ is even and } \left\lfloor \frac{n+1}{2} \right\rfloor \text{ is even} \\ c \oplus 1 & \text{if } n+1 \text{ is odd and } \left\lfloor \frac{n+1}{2} \right\rfloor \text{ is odd} \\ c & \text{if } n+1 \text{ is odd and } \left\lfloor \frac{n+1}{2} \right\rfloor \text{ is even,} \end{cases}$$

τ can be $2^{-(n+1)}$-approximated in $\mathrm{AC}^0 \Rightarrow \mathrm{PARITY} \in \mathrm{AC}^0$.

So by [Ajtai 1980], [Furst, Saxe, Sipser 1984], τ cannot be $2^{-O(n)}$-approximated in AC^0

Nowhere Differentiability in AC^0

Modify Takagi's function:

$$T(x) = \sum_{k=0}^{\infty} \frac{1}{2^{2^k-1}} \langle\langle 2^{2^k-1} x \rangle\rangle.$$

$2^{-O(n)}$-approximation: sum of $\lceil \log(n+1) \rceil + 2$ numbers of $2n+2$ bits – can be done in AC^0 ($\Rightarrow$ continuous)

Nowhere-differentiability: letting $\varphi_k(x) := \frac{1}{2^{2^k-1}} \langle\langle 2^{2^k-1} x \rangle\rangle$

$$T \text{ differentiable at } x \in [0,1] \Rightarrow T'(x) = \sum_{k=0}^{\infty} \varphi'_k(x)$$

However, this sum diverges since for all k, $|\varphi'_k(x)| = 1$.

ASIDE: bears a resemblence to a continuous nowhere-differentiable function defined by John McCarthy (1953)!

McCarthy's Function

AN EVERYWHERE CONTINUOUS NOWHERE DIFFERENTIABLE FUNCTION

JOHN MCCARTHY, Princeton University

The following is an especially simple example. It is

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} g(2^{2^n} x)$$

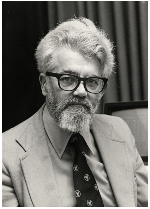
where $g(x) = 1 + x$ for $-2 \leqq x \leqq 0$, $g(x) = 1 - x$ for $0 \leqq x \leqq 2$ and $g(x)$ has period 4.

The function $f(x)$ is continuous because it is the uniform limit of continuous functions. To show that it is not differentiable, take $\Delta x = \pm 2^{-2^k}$, choosing whichever sign makes x and $x + \Delta x$ be on the same linear segment of $g(2^{2^k} x)$. We have

1. $\Delta g(2^{2^n} x) = 0$ for $n > k$, since $g(2^{2^n} x)$ has period $4 \cdot 2^{-2^n}$.
2. $\left| \Delta g(2^{2^k} x) \right| = 1$.
3. $\left| \Delta \sum_{n=1}^{k-1} 2^{-n} g(2^{2^n} x) \right| \leqq (k-1) \max \left| \Delta g(2^{2^n} x) \right| \leqq (k-1) 2^{2^{k-1}} 2^{-2^k} < 2^k 2^{-2^{k-1}}$.

Hence $\left| \Delta f / \Delta x \right| \geqq 2^{-k} 2^{2^k} - 2^k 2^{2^{k-1}}$ which goes to infinity with k.

The proof that the present example has the required property is simpler than that for any other example the author has seen.



J. McCarthy 1927-2011

Conclusion

Boolean approximation allows very low-complexity approximations for nowhere-differentiable functions

Consider further restrictions?

- Circuits of constant depth $k \in \mathbb{N}$ for some small k (but [Siu, Roychowdhury 1992] give depth-2 threshold circuits for iterated addition)
- Boolean (threshold) formulas
- Monotone

Generally, can we obtain some form of Boolean approximation for analytic functions that admits an inverse theorem?

More generally: theory of Boolean approximation of continuous functions