# Analysing Turing degree over intuititionistic logic

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# Intuitionistic logic

## Intuitionistic logic

- Reasoning by contradiction is not allowed.
- In general, we cannot use non-constructive principles such as

$$\begin{aligned} \text{LEM:}\ & A \vee \neg A; \\ \text{DNE:}\ & \neg\neg A \to A \text{ and} \\ \text{GDM:}\ & \neg \forall i < n A(i) \to \exists i < n \neg A(i), \end{aligned}$$

because they are proved by contradiction.

# Formal systems $\mathbf{i}\boldsymbol{\Delta}_0(\exp)$ and $\mathbf{i}\boldsymbol{\Sigma}_1$

- Let $L_1$ be the language of first order arithmetic consisting of $0,\ 1,\ +,\ \cdot,\ \exp,\ <$.
- The class $\Delta_0$ consists of formulae without unbounded quantifier. The classes $\Sigma_n$ and $\Pi_n$ are defined as usual.
- The system $\mathbf{i}\boldsymbol{\Delta}_0(\exp)$ consists of the following axioms:
  - Basic arithmetic

$$
\begin{array}{ll}
x + 0 = x; & x + (y+1) = (x+y) + 1; \\
x \cdot 0 = 0; & x \cdot (y+1) = x \cdot y + x; \\
\exp(x, 0) = 1; & \exp(x, y+1) = \exp(x, y) \cdot y; \\
\neg(x < x); & x < y \wedge y < z \to x < z; \\
x < x + 1; & x < y \to (x + 1 < y) \vee (x + 1 = y).
\end{array}
$$

  - $\Delta_0$-IND

$$B(0) \wedge \forall x(B(x) \to B(x+1)) \to \forall x B(x), \text{for } B \in \Sigma_1.$$

- $\mathbf{i}\boldsymbol{\Sigma}_1 \equiv \mathbf{i}\boldsymbol{\Delta}_0(\exp) + \Sigma_1\text{-IND}.$

# What has been done?

Main reference:

Troelstra and van Dalen, *Constructivism in Mathematics I* (1988) [7],

1. Definition of partial recursive functions
2. Enumeration theorem: There is a recursive function enumerating all $n$-ary partial recursive functions, i.e., there is an $(n+1)$-ary partial rec. function $\varphi_z$ s.t. $\varphi_z(e, x_1, ..., x_n) \simeq \varphi_e(x_1, ..., x_n)$.
3. Normal form theorem: There is a recursive predicate $T$ and primitive recursive function $U$ such that $\varphi_e(x) = y \leftrightarrow U(\min z(Texz)) = y$. (so-called Kleene's $T$ predicate and result-extracting function $U$)
4. smn theorem: There is a primitive recursive $s$ such that

$$\forall x_1, ...x_m, y_1, ..., y_n(\varphi_e(x_1, ..., x_m, y_1, ...y_n) \simeq \varphi_{s(e,x_1,...x_m)}(y_1, ..., y_n))$$

5. Recursion Theorem: For each recursive $f$, there is $n$ such that $\forall x(\varphi_n(x) \simeq \varphi_{f(n)}(x))$.
6. Formalization in $\mathbf{HA}$ and $\mathbf{EL}$ (needs only $\Sigma_1$ induction).
   ②-⑤ can be parameterized.

# Recursively enumerable set and its property

## Definition (Recursively enumerable set)

Let $W_e = \{x : \{e\}(x)\downarrow\} = \{x : \varphi_e(x)\downarrow\}$.
A set $A$ is *recursively enumerable* (r.e.) if $A = W_e$ for some $e$.

For a set $A \subseteq \mathbb{N}$, the *characteristic function* of $A$ is a *total* function $\chi_A : \mathbb{N} \to \{0, 1\}$ s.t.

$$\chi_A(x) = \begin{cases} 0 & \text{if } x \in A; \\ 1 & \text{if } x \notin A. \end{cases}$$

## Note

If we have the characteristic function $\chi_A$ for $A$, then we have

$$\forall x(x \in A \vee x \notin A).$$

## Definition (Recursive set)

A set $A$ is *recursive* if there is $e$ s.t. $\chi_A = \{e\}$.

# Oracles in computation

- For $\sigma \in 2^{<\omega}$, $\sigma \subset A$ means $\sigma(i) = \chi_A(i)$ for all $i < lh(\sigma)$.
- $\{e\}^A_s(x) = y$ if $x, y, e < s$, $s > 0$, $\{e\}^A(x) = y$ in $< s$ steps according to the program coded by $e$, and only numbers $z < s$ are used in the computation.
- The *use function*

$$u(A; e, x, s) = \begin{cases} 1 + \begin{array}{l}\text{the maximal } n \text{ used in} \\ \text{the computation of } \{e\}^A(x)\end{array} & \text{if } \{e\}^A_s(x)\downarrow; \\ 0 & \text{otherwise.} \end{cases}$$

- $\{e\}^\sigma_s(x) = y$ if $\{e\}^A_s(x) = y$ for some $A \supset \sigma$, and only $z < lh(\sigma)$ is used in the computation.

## Use Principle

1. $\{e\}^A(x) = y \Longrightarrow \exists s \exists \sigma \subset A(\{e\}^\sigma_s(x) = y)$
2. $\{e\}^\sigma_s(x) = y \Longrightarrow \forall t \geq s \forall \tau \supseteq \sigma(\{e\}^\tau_t(x) = y)$
3. $\{e\}^\sigma_s(x) = y \Longrightarrow \forall A \supset \sigma(\{e\}^A(x) = y)$

Let $W^A_{e,s} = \{x : \{e\}^A_s(x)\downarrow\}$ and $W_{e,s} = W^\emptyset_{e,s}$.

# Turing reducibility

**Definition (Turing reducibility)**

$$A \leq_T B \iff \text{there is } e \text{ s.t. } \forall x(\chi_A(x) = \{e\}(\chi_B, x) \stackrel{\text{def}}{=} \{e\}^B(x))$$

**Definition (recursively enumerable in $A$)**

$B$ is r.e. in $A$ is $B = W_e^A = \{x : \{e\}^A(x)\downarrow\}$ for some $e$.

# Turing jump

## Definition (Turing jump)

$$A' = \{x : \{x\}^A(x)\downarrow\}$$

## Jump Theorem

1. $A' \not\leq_T A$
2. $B \leq_T A$ iff $B \leq_1^s A$, i.e., there is a total rec. function $f$ s.t.

$$x \in B' \leftrightarrow f(x) \in A'.$$

## Post's problem

Is there any $A$ s.t. $\emptyset \lneq_T A \lneq_T \emptyset'$?

## Definition (Low set)

A set $A \leq_T \emptyset'$ is *low* if $A' \equiv_T \emptyset'$, i.e., $A \leq_T \emptyset'$ and $\emptyset' \leq_T A'$.

# Oracles and law of excluded middle

## Lemma

For any r.e. $A$, there is a total rec. function $f$ s.t.

$$\forall x(x \in A \leftrightarrow f(x) \in \emptyset')$$

## Lemma in other words

For any $\Sigma_1$ formula $A(x)$ without set parameter, there is a total rec. function $f$ s.t.

$$\forall x(A(x) \leftrightarrow \chi_{\emptyset'}(f(x)) = 0)$$

In particular, if we have $\chi_{\emptyset'}$, then $A \vee \neg A$ holds for any $\Sigma_1$ formula $A$ without set parameters.
In general, if we have $\chi_B$ and $\{x : A(x)\} \leq_T \{x : B(x)\}$, then we have $\forall x(A(x) \vee \neg A(x))$.

# Classical definitions simple sets and immune sets

## Definition (Immune set and simple set)

- $A$ is *immune* if
  - $A$ is infinite;
  - $\forall e(W_e$ is infinite $\rightarrow W_e \not\subseteq A)$,
- $A$ is *simple* if
  - $A$ is r.e.;
  - $\overline{A} = \{x : x \notin A\}$ is immune.

# Post's construction of a simple set

**Theorem (Post)**

There is a simple set $A$.

*Proof.* Define sets $A$ and $S$ and a partial rec. $\psi$.

- $A = \{(e, x) : x \in W_e \wedge x > 2e\}$;
- $\psi(e) = x$ iff $\exists u(Texu \wedge U(u) = x \wedge x > 2e)$;
- $S = \mathrm{rang}(f)$.

Even constructively, we can prove the following holds.

1. $A$ and $S$ are r.e.
2. $\forall x \neg\neg\exists y > x(x \notin S)$.
   For each $e$ s.t. $\psi(e)\downarrow$, $\psi(e) > 2e$. Therefore
   $|\{0, 1, \ldots, 2e\} \cap S| = |\{0, 1, \ldots, 2e\} \cap \{\psi(i) : \psi(i)\downarrow \wedge i < e\}| < e$
3. If $\forall x \exists y > x(y \in W_e)$, then $\exists y(y \in S \cap W_e)$.

# Constructive observation

## What is a reasonable definition for immne/simple sets?

- $A$ is *weakly immune* if
  - $\forall x \neg\neg\exists y > x(y \in A)$;
  - $\forall e[\forall x \exists y > x(y \in W_e) \to \neg\neg\exists y \in W_e(y \notin A)]$,
- $A$ is *weakly simple* if
  - $A$ is r.e.;
  - $\overline{A} = \{x : x \notin A\}$ is weakly immune.
- $A$ is *immune* if
  - $\forall x \exists y > x(y \in A)$;
  - $\forall e[\forall x \exists y > x(y \in W_e) \to \neg\neg\exists y \in W_e(y \notin A)]$,
- $A$ is *simple* if
  - $A$ is r.e.;
  - $\forall x \exists y > x(y \notin A)$;
  - $\forall e[\forall x \exists y > x(y \in W_e) \to \exists y(y \in W_e \cap A)]$.

## Semi constructive results

- $\mathbf{i\Delta_0}(\exp) \vdash$ "There is a weakly simple set $A$".
- $\mathbf{i\Delta_0}(\exp) + \Sigma_1^0\text{-GDM} \vdash$ "There is a simple set $A$".

# Take a close look to classical finite injury method

**Theorem (Friedberg-Muchnik)**

There is a low simple set $A$.

For such $A$,

- By simplicity, $A$ is r.e. but not recursive.
- By lowness, $A' \leq_T \emptyset'$ and so $A \lneq_T \emptyset'$.

# Take a close look to classical finite injury method

**Theorem (Friedberg-Muchnik)**

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*Proof.*

Construct $A = \bigcup A_s$ as follows. Every $A_s$ is recursive and so $A$ is r.e.:

Stage $s = 0$ $A_0 = \emptyset$.

Stage $s + 1$ Choose the least $i \leq s$ s.t.

1. $W_{i,s} \cap A_s = \emptyset$;
2. $\exists x[x \in W_{i,s} \wedge x > 2i \wedge (\forall e \leq i)[u(A_s; e, e, s) < x]]$.

If such $i$ exists, choose the least $x$ satisfying ② and let $A_{s+1} = A_s \cup \{x\}$. If such $i$ does not exists, let $A_{s+1} = A_s$.

Construct $A = \bigcup A_s$ as follows:

Stage $s = 0$ $A_0 = \emptyset$.

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If such $i$ exists, choose the least $x$ satisfying ② and let $A_{s+1} = A_s \cup \{x\}$. If such $i$ does not exists, let $A_{s+1} = A_s$.

Classical proof

$P_e$: $\forall x \exists y > x(x \in W_e) \rightarrow \exists x(x \in W_e \cap A)$

$N_e$: $\exists x(\lim_{s\to\infty}\{e\}_s^{A_s}(e) = x)$.

Construct $A = \bigcup A_s$ as follows:

Stage $s = 0$ $A_0 = \emptyset$.

Stage $s+1$ Choose the least $i \leq s$ s.t.

1. $W_{i,s} \cap A_s = \emptyset$;
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If such $i$ exists, choose the least $x$ satisfying ② and let $A_{s+1} = A_s \cup \{x\}$. If such $i$ does not exists, let $A_{s+1} = A_s$.

Classical proof

$P_e$: $\forall x \exists y > x(x \in W_e) \rightarrow \exists x(x \in W_e \cap A)$

$N_e$: $\exists x(\lim_{s\rightarrow\infty}\{e\}_s^{A_s}(e) = x)$.

- $\forall e P_e$ guarantees simplicity.
  - $|\{0, 1, \ldots, 2e\} \cap A| \leq e$, which implies $\forall e \exists B \subset \overline{A}(|B| \geq e)$, and so $\forall x \exists y > x(x \in \overline{A})$.

Construct $A = \bigcup A_s$ as follows:

Stage $s = 0$ $A_0 = \emptyset$.

Stage $s + 1$ Choose the least $i \leq s$ s.t.

1. $W_{i,s} \cap A_s = \emptyset$;
2. $\exists x[x \in W_{i,s} \wedge x > 2i \wedge (\forall j \leq i)[u(A_s; j, j, s) < x]]$.

If such $i$ exists, choose the least $x$ satisfying ② and let $A_{s+1} = A_s \cup \{x\}$. If such $i$ does not exists, let $A_{s+1} = A_s$.

Classical proof

$P_e$: $\forall x \exists y > x(x \in W_e) \to \exists x(x \in W_e \cap A)$

$N_e$: $\exists x(\lim_{s\to\infty}\{e\}_s^{A_s}(e) = x)$.

- $\forall e P_e$ guarantees simplicity.
  - $|\{0, 1, \ldots, 2e\} \cap A| \leq e$, which implies $\forall e \exists B \subset \overline{A}(|B| \geq e)$, and so $\forall x \exists y > x(x \in \overline{A})$.
- $\forall e N_e$ guarantees lowness.
  - $g(e, s) = \begin{cases} 1 & \text{if } \{e\}_s^{A_s}(e)\downarrow; \\ 0 & \text{otherwise.} \end{cases}$ is recursive and $\forall e N_e$ implies $\lim_{s\to\infty} g(e, s) = \hat{g}(e)$ exists for all $e$. Then $\hat{g} = \chi_{A'}$ and $\hat{g} \leq_T \emptyset'$.

Construct $A = \bigcup A_s$ as follows:

**Stage $s = 0$** $A_0 = \emptyset$.

**Stage $s+1$** Choose the least $i \leq s$ s.t.

1. $W_{i,s} \cap A_s = \emptyset$;
2. $\exists x[x \in W_{i,s} \wedge x > 2i \wedge (\forall j \leq i)[u(A_s; j, j, s) < x]]$.

If such $i$ exists, choose the least $x$ satisfying ② and let $A_{s+1} = A_s \cup \{x\}$. If such $i$ does not exists, let $A_{s+1} = A_s$.

Classical proof

$P_e$: $\forall x \exists y > x(x \in W_e) \to \exists x(x \in W_e \cap A)$ (simplicity)

$N_e$: $\exists x(\lim_{s\to\infty} \{e\}_s^{A_s}(e) = x)$. (lowness)

How to prove $\forall e P_e$ and $\forall e N_e$?

($\forall e N_e$) Since $I_e = \{s : \{e\}_s^{A_s}(e) \not\simeq \{e\}_{s+1}^{A_{s+1}}(e)\}$ satisfies $|I_e| \leq 2e$, $\hat{u}(e) = \lim_{s\to\infty} u(A_s; e, e, s)$ exists for each $e$.

($\forall e P_e$) By induction, $\max\{\hat{u}(i) : i \leq e\}$ is defined for each $e$. If $\forall x \exists y > x(y \in W_e)$, there is $s_e$ s.t. $\exists x(x \in W_{e,s_e} \cap A_s)$.

# What non-constructive principle are sufficient?

$P_e$: $\forall x \exists y > x (x \in W_e) \to \exists x (x \in W_e \cap A)$

$N_e$: $\exists x (\lim_{s \to \infty} \{e\}_s^{A_s}(e) = x)$.

- $\forall e P_e$ guarantees simplicity.
  - $|\{0, 1, \dots, 2e\} \cap A| \leq e$, which implies $\forall x \neg\neg \exists y > x (y \notin A)$.
- $\forall e N_e$ guarantees lowness.
  - $g(e, s) = \begin{cases} 1 & \text{if } \{e\}_s^{A_s}(e)\downarrow; \\ 0 & \text{otherwise.} \end{cases}$ is recursive.

    $\forall e N_e$ implies $\lim_{s \to \infty} g(e, s) = \hat{g}(e)$ exists for all $e$.
    Then $\hat{g} = \chi_{A'}$ and $\hat{g} \leq_T \emptyset'$.

How to prove $\forall e P_e$ and $\forall e N_e$?

($\forall e N_e$) Since $I_e = \{s : \{e\}_s^{A_s}(e) \not\simeq \{e\}_{s+1}^{A_{s+1}}(e)\}$ satisfies $|I_e| \leq 2e$, by $\Sigma_1^0$-BCA, $\hat{u}(e) = \lim_{s \to \infty} u(A_s; e, e, s)$ exists for each $e$.

($\forall e P_e$) By $\Sigma_1^0$-BCA, $\max\{\hat{u}(i) : i \leq e\}$ is defined for each $e$.
If $\forall x \exists y > x (y \in W_e)$, there is $s_e$ s.t. $\exists x (x \in W_{e,s_e} \cap A_s)$.

## Logical principles used

$\Sigma_1$-BCA $\forall e \exists t \in \{0,1\}^e (\forall i < e)[t(i) = 0 \leftrightarrow A(x)]$ for $A(x) \in \Sigma_1$.

## Decomposition of $\Sigma^0_1$-BCA

The following are equivalent over $\mathbf{i\Delta_0}(\exp)$.

1. $\Sigma_1$-BCA;
2. $\Delta(\Sigma_1)$-IND + $\Delta(\Sigma_1)$-LEM;
3. $\Delta(\Sigma_1)$-IND + $\Sigma_1$-LEM;
4. $\Sigma_1$-IND + $\Delta(\Sigma_1)$-LEM;

where $\Delta(\Sigma_1)$ is the smallest class containing $\Sigma_1$ and closed under $\wedge, \vee, \rightarrow, \forall x < t, \exists x < t$ and $\Gamma$-LEM is $A \vee \neg A$ for $A \in \Gamma$.

## How BCA works?

The following are equivalent over $\mathbf{i\Delta_0}(\exp)$:

- $\Sigma_1$-BCA;
- For any $A \in \Sigma_1$, $\exists x(|A| = x) \rightarrow \exists x \forall y \in A(y < x)$;
- For any $B \in \Delta_1$, $\forall e \exists x \forall i < e(\exists y B(i,y) \rightarrow \exists y < x B(i,y))$.

# Constructive observation

The construction of r.e. $A$ needs no non-constructive principles.

**Stage $s = 0$** $A_0 = \emptyset$.

**Stage $s + 1$** Choose the least $i \leq s$ s.t.

1. $W_{i,s} \cap A_s = \emptyset$;
2. $\exists x[x \in W_{i,s} \wedge x > 2i \wedge (\forall j \leq i)[u(A_s; j, j, s) < x]]$.

## Weaken non-constructive principle as much as possible

There is a $B(x) \in \Sigma_1$ satisfying the following:

- $\mathbf{i\Delta_0}(\exp) \vdash \forall x \exists s {\in} \{0,1\}^x \forall i{<}x (s_i{=}0 \leftrightarrow B(i)) \rightarrow$ "$A$ is a low and simple set".
- $\mathbf{i\Delta_0}(\exp) \vdash \forall x \exists s {\in} \{0,1\}^x \forall i{<}x (s_i{=}0 \leftrightarrow B(i)) \rightarrow \neg\exists e(\chi_{\emptyset'}{=}\{e\}^A)$
- $\mathbf{i\Delta_0}(\exp){+}\Delta(\Sigma_1)\text{-IND} \vdash \forall x(B(x) \vee \neg B(x)) \rightarrow \neg\exists e(\chi_{\emptyset'}{=}\{e\}^A)$
- $\mathbf{i\Delta_0}(\exp){+}\Delta(\Sigma_1)\text{-IND} \vdash \neg\neg\forall x(B(x) \vee \neg B(x)) \rightarrow \neg\exists e(\chi_{\emptyset'}{=}\{e\}^A)$

## Theorem

$\mathbf{i\Delta_0}(\exp){+}\Delta(\Sigma_1)\text{-IND}{+}\neg\neg(\Sigma_1\text{-LEM}) \vdash \neg\exists e(\chi_{\emptyset'}{=}\{e\}^A)$,
where $\neg\neg(\Gamma\text{-LEM})$ is $\neg\neg\forall x(B(x) \vee \neg B(x))$ for $B \in \Gamma$.

# Constructive observation

The construction of r.e. $A$ needs no non-constructive principles.

Stage $s = 0$ $A_0 = \emptyset$.

Stage $s+1$ Choose the least $i \leq s$ s.t.

1. $W_{i,s} \cap A_s = \emptyset$;
2. $\exists x[x \in W_{i,s} \wedge x > 2i \wedge (\forall j \leq i)[u(A_s; j, j, s) < x]]$.

**Lemma**

$\mathbf{i\Delta_0}(\exp) + \Pi_1\text{-IND} \vdash \forall x \neg\neg \exists s {\in} \{0,1\}^x \forall i{<}x (s_i{=}0 \leftrightarrow B(i))$, for $B \in \Sigma_1$.

**Weaken induction as much as possible**

There is a $B(x) \in \Sigma_1$ satisfying the following:

- $\mathbf{i\Delta_0}(\exp) \vdash \forall x \exists s {\in} \{0,1\}^x \forall i{<}x (s_i{=}0 \leftrightarrow B(i)) \rightarrow$ "$A$ is a low and simple set".
- $\mathbf{i\Delta_0}(\exp) \vdash \neg\neg\forall x \exists s {\in} \{0,1\}^x \forall i{<}x (s_i{=}0 \leftrightarrow B(i)) \rightarrow \neg\exists e(\chi_{\emptyset'}{=}\{e\}^A)$

**Theorem**

$\mathbf{i\Delta_0}(\exp) + \Pi_1\text{-IND} + \Delta(\Sigma_1)\text{-DNS} \vdash \neg\exists e(\chi_{\emptyset'=\{e\}^A})$,
where $\Gamma$-DNS is $\forall x \neg\neg C(x) \rightarrow \neg\neg\forall x C(x)$ for $C \in \Gamma$.

# A classical construction of two incomparable degrees

Define two sequences $\{f_s\}_s$ and $\{g_s\}_s$ of $2^{<\mathbb{N}}$ as follows:

1. For $s = 0$, let $f_0 = g_0 = \langle\rangle$.
2. At stage $s = 2e + 1$: Let $n = lh(f_s)$.
   1. If $\exists t \exists \sigma(\sigma \supsetneq g_s \wedge \{e\}^{\sigma}_{t}(n)\downarrow)$, take the least such pair $(\sigma, t)$ and let
   $$g_{s+1} = \sigma, \qquad f_{s+1} = f_s * \langle 1 \dot{-} \{e\}^{\sigma}_{t}(n)\rangle.$$
   2. If $\neg\exists t \exists \sigma(\sigma \supsetneq g_s \wedge \{e\}^{\sigma}_{t}(n)\downarrow)$, set
   $$g_{s+1} = g_s * \langle 0\rangle, \qquad f_{s+1} = f_s * \langle 0\rangle.$$
3. At stage $s = 2e + 2$: Do the same but with the roles of $f_s$ and $g_s$ are interchanged.

By $\Sigma_2$-IND, each $f_s$ and $g_s$ are defined and have length $\geq s$, $f_s \subseteq f_{s+1}$ and $g_s \subseteq g_{s+1}$.
By the construction, $f = \bigcup_s f_s$ and $g = \bigcup_s g_s$ satisfy

$$f(lh(f_s)) \neq \{e\}^{g}(lh(f_s) \qquad g(lh(g_{s+1})) \neq \{e\}^{f}(lh(g_{s+1}))$$

for each $e = 2s + 1$.

# Constructive observation

The previous proof shows that

$$\forall e \forall s \forall n (\exists t \exists \sigma \supsetneq s \{e\}_t^\sigma(n)\downarrow) \vee \neg \exists t \exists \sigma \supsetneq s(\{e\}_t^\sigma(n)\downarrow) \to$$
"there are two incomparable degrees below $\emptyset'$"

## Theorem

$\mathbf{i\Sigma_2} + \Sigma_1\text{-LEM} \vdash$ "there are two incomparable degrees below $\emptyset'$".

# Some observation and future work

- $\forall x \neg\neg\exists y > x(y \in A)$ is a weak notion of infiniteness. $\forall x \neg\neg\exists y > x(y \in A) \to \forall x \exists y > x(y \in A)$ for $A \in \Sigma_1$ is equivalent to $\Sigma_1$-DNE.
- This seems to be a refinement of the result by Groszek and Slaman [2], which states that Friedberg-Muchnik construction can be formalized in $\mathrm{I}\Sigma_1$.
- In [6], it is shown that $A$ s.t. $\emptyset \lneq_T A \lneq_T \emptyset'$ exists in every model of $\mathrm{I}\Delta_0(\exp)$.
  What non-constructive principle is need for it?
- Considering complement in constructive setting, Bishop [1] suggested the notion of complemented set:
  - Complemented set is a pair $A = (A_0, A_1)$ of sets $A_i \subseteq \mathbb{N}$ s.t. each $x \in A_0$ and $y \in A_1$ there is $f : \mathbb{N} \to \mathbb{R}$ with $f(x) \neq f(y)$.
  - $x \in A$ means $x \in A_0$ and $x \notin A$ means $x \in A_1$.

  This is a way to treat $x \notin A$ in a positive way.
  It might be useful in our setting, but I have not yet found such examples.
- And much more things to do!

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