A Verified Power-Series Method for Multivariate IVPs

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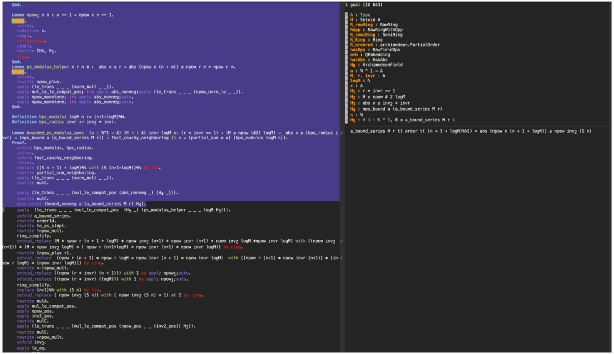
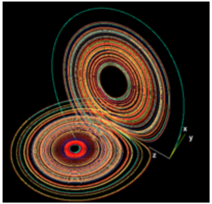
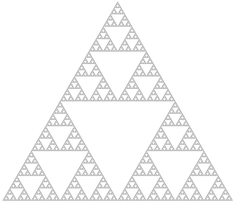
Computation and Formal Proof

A **proof assistant** is an interactive tool for writing and checking formal proofs.







Ordinary Differential Equations (ODEs)

In this talk we consider initial value problems (IVPs) for ordinary differential equations (ODEs). An IVP is an ODE together with an initial condition at

$$\dot{\vec{y}}(t) = \vec{F}(t, \vec{y}(t)), \quad \vec{y}(t_0) = \vec{y}_0, \quad \vec{F} : D \subseteq \mathbb{R}^{d+1} \to \mathbb{R}^d.$$

A solution is a differentiable function $\vec{y} : I \to \mathbb{R}^d$, where $I \subseteq \mathbb{R}$ is an open interval containing t_0, that satisfies the IVP.

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Remarks:

- Higher-order ODEs $\vec{y}^{(k)} = \vec{G}(t, \vec{y}, \dots, \vec{y}^{(k-1)})$ can be rewritten as a first-order ODEs by introducing additional variables for the higher derivatives $\vec{y}^{(2)}, \dots, \vec{y}^{(k)}$.
- Any ODE can be turned into an equivalent autonomous ODE by adding t as additional variable.
- By translation, one can assume $t_0 = 0$ and $\vec{y}_0 = \vec{0}$ without loss of generality.

ODE solving in Proof Assistants

The problem is a natural candidate for formal verification:

- It is a classical topic in analysis with many applications in science and engineering.
- Most ODEs do not have a closed-form solution, therefore numerical methods are required to approximate solutions.
- Guaranteeing correctness of these approximations is important for safety-critical applications.

ODE solving in Rocq

Although other proof assistants (e.g. Isabelle/HOL) include substantial formalizations of ODE theory and numerical methods, not so much is available in Rocq so far.

- The CoRN library includes an implementation of the Picard Iteration method. [1]

[1] Makarov & Spitters. *The Picard algorithm for ordinary differential equations in Coq*, Proc. of ITP 2013.
[2] Park & T. *A Coq Formalization of Taylor Models and Power Series for Solving ODEs*, Proc. of ITP 2024.

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Since Rocq is based on constructive logic, a constructive/computable formalization is natural.

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Computability and Complexity of ODEs

Assume $f : [0,1] \times [-1,1] \to \mathbb{R}$ and consider the IVP $\dot{y}(t) = f(t, y(t)); y(0) = 0$

Assumptions on f	**Computability/Complexity of Solution**
(Polynomial-time) computable	All solutions can be non-computable [3]
+ Unique solution	Computable, but complexity can be arbitrarily high [4]
+ Lipschitz continuous	PSPACE complete [5]
+ Analytic	Solution is polynomial-time computable [6]

[3] Aberth, *The failure in computable analysis of a classical existence theorem for differential equations*, 1971.
[4] Ko, *On the computational complexity of ordinary differential equations*, Journal of Computer and System Sciences, 1983.
[5] Kawamura, *Lipschitz Continuous Ordinary Differential Equations are Polynomial-Space Complete*, Comp. Complexity, 2010.
[6] Müller & Moiske, *Solving initial value problems in polynomial time*, Proc. of JAIIO — Panel '93, 1993.

The Cauchy-Kovalevskaya Theorem

Theorem (ODE version of the Cauchy-Kovalevskaya Theorem)
Let $U \subseteq \mathbb{R}^d$ be open and $f : U \to \mathbb{R}^d$ analytic. Then the initial value problem

$$\dot{\vec{y}}(t) = \vec{F}(\vec{y}(t)), \quad \vec{y}(0) = \vec{y}_0$$

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Main idea of the classical proof:

- Expand $\vec{F}(\vec{y})$ in a formal power series around $\vec{y_0}$.
- Derive a recurrence for the Taylor coefficients of $\vec{y}(t)$ from the ODE $\dot{\vec{y}} = \vec{F}(\vec{y})$.
- Prove that the formal series has a positive radius of convergence by dominating it by a geometric majorant sequence.

Formal Power Series Solution

Let

$$\dot{\vec{y}}(t) = \vec{F}(\vec{y}(t)), \quad \vec{y}(0) = \vec{y}_0.$$

We inductively define a sequence of functions $\vec{F}^{[n]} : \mathbb{R}^d \to \mathbb{R}^d$ of multivariate functions by

$$\vec{F}^{[0]}(\vec{x}) = \vec{x}, \text{ and}$$

$$\vec{F}^{[n+1]}(\vec{x}) = J_{\vec{F}^{[n]}}(\vec{x}) \cdot \vec{f}(\vec{x})$$

Define a function by the formal power series

$$\vec{y}(t) = \sum_{i=0}^{\infty} \frac{1}{n!} \vec{F}^{[n]}(\vec{y_0}) t^n$$

By deriving $\vec{y}$ (as formal power series) and comparing coefficients, we see that $\vec{y}$ satisfies the IVP.

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$$\begin{aligned} \dot{y}(t) &= F(y(t)) \\ \ddot{y}(t) &= \dot{y}(t) \cdot F'(y(t)) \\ &= F(y(t)) \cdot F'(y(t)) \\ &\vdots \end{aligned}$$

The Method of Majorants

- A (single-variate) power series $M(x) = \sum_{k=0}^{\infty} A_k x^k$ is said to majorize a d-variate power series $f(x) = \sum_{\alpha \in \mathbb{N}^d} a_\alpha x^\alpha$ if

$$\forall k \in \mathbb{N}, \quad \sum_{|\alpha|=k} |a_\alpha| \ \leq \ A_k, \quad \text{where } |\alpha| = \sum_{i=1}^{d} \alpha_i.$$

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- We are mostly interested in simple (geometric) majorants of the form $\sum_{k=0}^{\infty} MR^k$ for constant $M, R \in \mathbb{R}$.
- In that case, the tail of the series satisfies

$$\Big| \sum_{|\alpha|>k} a_\alpha \, x^\alpha \Big| \ \le \ M \sum_{n=k+1}^{\infty} (R\|x\|)^n \ = \ M \, \frac{(R\|x\|)^{k+1}}{1 - R\|x\|}, \quad (R\|x\| < 1).$$

Solution Majorants

- We can show that if the right-hand side function $f : \mathbb{R}^d \to \mathbb{R}^d$ has a geometric majorant (M, R) then $(1, 2dMR)$ works as a geometric majorant for the solution y, guaranteeing convergence of $y(t)$ for all t with $|t| < \frac{1}{2dMR}$.

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- On the other hand for

$$\dot{y}(t) = \frac{1}{1 - y(t)} = \sum_{k=0}^{\infty} y(t)^k \; ; \; y(0) = 0$$

we get radius $\frac{1}{2}$ which is identical to the actual radius of the solution

$$y(t) = 1 - \sqrt{1 - 2t}.$$

Polynomial IVPs

- ODEs with polynomial right-hand side are an important special case.
- Polynomials allow efficient implementations of evaluation and other operations.
- As the coefficient sequence is finite, majorant bounds can be computed automatically.
- Many analytic ODEs (including all systems built from elementary functions) can be rewritten as polynomial ODEs by increasing the dimension.

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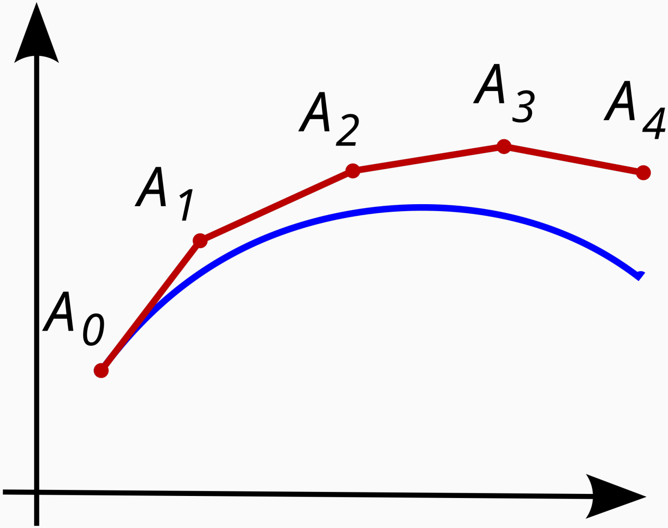
Example: $y_1' = \frac{1}{2y_1};\ y_1(0) = 1$ The IVP has the solution $y_1(x) = \sqrt{x+1}$.

Introduce auxiliary variable $y_2 = \frac{1}{y_1}$ to obtain:

$$\begin{aligned} \dot{y}_1 &= \tfrac{1}{2}\, y_2, & y_1(0) &= 1, \\ \dot{y}_2 &= -\tfrac{1}{2}\, y_2^3, & y_2(0) &= 1. \end{aligned}$$

Extending the solution



Demo

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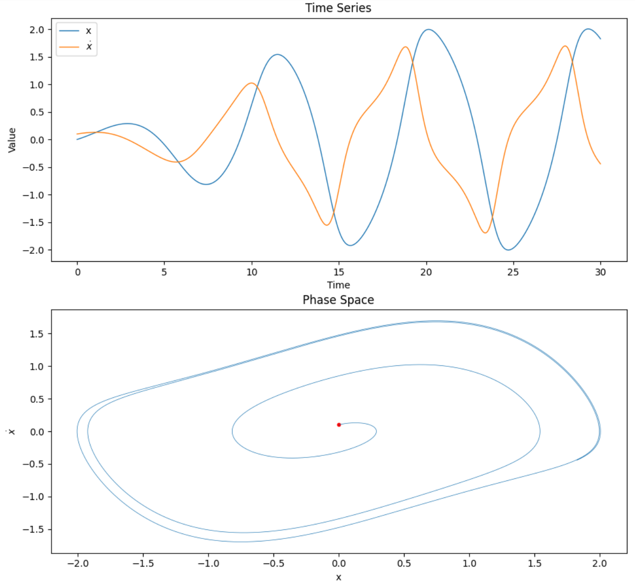

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- Allows to compute interval trajectories inside the proof assistant.
- Currently only very crude error bounds, sharpening would improve efficiency.
- In general, optimizing computation inside the Rocq proof assistant is challenging.

Thank you!