Computable transformations of Turing degrees

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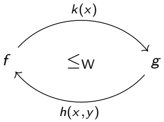
Weihrauch reducibility

Problems:

Partial, multivalued functions $f :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$

Reduction:

A problem $f :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ reduces to $g :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$
via programs $h, k :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$:



- For each instance $x \in \mathrm{dom}(f)$, the program $k(x)$ terminates and produces an instance $k(x) \in \mathrm{dom}(g)$
- For each instance $x \in \mathrm{dom}(f)$ and solution $y \in (g \circ k)(x)$, the program $h(x, y)$ terminates and produces a solution $h(x, y) \in f(x)$.

Compositional product

Concatenation

Given two problems $f, g :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$, we define $f \circ g :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ with

$$\mathrm{dom}(f \circ g) := \{x \in \mathrm{dom}(g) \mid g(x) \subseteq \mathrm{dom}(f)\}$$
$$(f \circ g)(x) := \{z \in \mathbb{N}^{\mathbb{N}} \mid z \in f(y) \text{ and } y \in g(x) \text{ for some } y \in \mathbb{N}^{\mathbb{N}}\}$$

Composition

Given two problems $f, g :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$, we consider the degree

$$f * g := \max_{\leq_{\mathrm{W}}}\{f' \circ g' \mid f' \leq_{\mathrm{W}} f \text{ and } g' \leq_{\mathrm{W}} g\}$$

Two applications of the Gödel problem

Two applications of the Gödel problem (1/5)

The Gödel problem is given by

$$\mathsf{G} :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}$$

with

$$\mathsf{G}(x) := \{e \in \mathbb{N} \mid x = \varphi_e\}$$

Does

$$\mathsf{G} * \mathsf{G} \leq_{\mathsf{W}} \mathsf{G}$$

hold?

(It does with respect to the continuous degrees, Brattka 2023)

Two applications of the Gödel problem (2/5)

Gödel problem:

$$G(x) := \{e \in \mathbb{N} \mid x = \varphi_e\}$$

Prove

$$G * G \not\leq_W G$$

Using the helper problem

$$H :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}$$

with

$$H(\langle x, y_0, \dots \rangle) := \{e \in \mathbb{N} \mid y_n = \varphi_e \text{ for some } n \in \mathbb{N} \text{ with } x = \varphi_n\}$$

(i.e. for any code n of x, produce a code of y_n)

this reduces to the claim

$$H \not\leq_W G$$

Two applications of the Gödel problem (3/5)

Assume, for contradiction

$$\mathsf{H} \leq_{\mathsf{W}} \mathsf{G}$$

realized by $h, k :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$. In particular: $\mathrm{rng}(k)$ is countable.

Proceed by case distinction on

Splitting

For any (nice) $\sigma \in 2^*$, there are extensions $\tau =: \langle \tau_0, \ldots \rangle, \tau' =: \langle \tau'_0, \ldots \rangle \in 2^*$ of σ with

- $\tau_0 = \tau'_0$,
- $\#(\tau_n) \leq 1$ and $\#(\tau'_n) \leq 1$ for each $n > 0$,
- φ_e^{τ} and $\varphi_e^{\tau'}$ incomparable (for fixed $e \in \mathbb{N}$ with "$k \subseteq \varphi_e$").

Two applications of the Gödel problem (4/5)

If "Splitting" fails:

$\Rightarrow$ There is a nice $\sigma = \langle \sigma_0, \ldots, \sigma_n \rangle$ (i.e. $\#(\sigma_i) \leq 1$ for $0 < i < n$) s.t. φ_e^τ and $\varphi_e^{\tau'}$ are comparable for any nice extensions τ, τ' of σ.

$\Rightarrow$ We have $k(x) = k(y)$ for nice extensions $x, y \in \mathrm{dom(H)}$.

- Choose $e' \in (\mathrm{G} \circ k)(\sigma * 000\ldots)$,
- Choose extension $x \in 2^{\mathbb{N}}$ of σ_0 with $\mathrm{G}(x) > n$,
- Define $y_i := \sigma_{i+1} * 000\ldots$ for all $i < n$.

$\Rightarrow$ $\langle x, y_0, \ldots, y_{n-1}, z, \ldots \rangle \in \mathrm{dom(H)}$ and
it extends σ for all $z \in 2^{\mathbb{N}}$ with $\#(z) \leq 1$

$$\mathrm{G}(z) = \mathrm{H}(\langle x, y_0, \ldots, y_{n-1}, z, \ldots \rangle)$$

$$h(\langle \langle x, y_0, \ldots, y_{n-1}, z, \ldots \rangle, e' \rangle) \in \mathrm{H}(\langle x, y_0, \ldots, y_{n-1}, z, \ldots \rangle) \quad \lightning$$

Two applications of the Gödel problem (5/5)

If “Splitting” holds, we define $T : 2^* \to 2^*$:

$$T(\langle 0,0\rangle) := \tau \qquad T(\langle 0,1\rangle) := \bar{\tau} \qquad T(\langle 1,0\rangle) := \tau' \qquad T(\langle 1,1\rangle) := \bar{\tau}'$$

$$T(\langle 0\rangle) := \sigma \qquad T(\langle 1\rangle) := \sigma'$$

$$T(\langle\rangle) := \langle\rangle$$

- $(\pi_0 \circ T)(0\dots) = (\pi_0 \circ T)(x)$ for all $x \in 2^{\mathbb{N}}$,
- $(\pi_{n+1} \circ T)(x)$ contains at most one 1 for all $n \in \mathbb{N}$, $x \in 2^{\mathbb{N}}$,
  $\implies (\pi_n \circ T)(x)$ computable for all $n \in \mathbb{N}$ and $x \in 2^{\mathbb{N}}$
  $\implies T(x) \in \mathrm{dom}(H)$ for all $x \in 2^{\mathbb{N}}$
- $(k \circ T)(x) \neq (k \circ T)(y)$ for all $x, y \in 2^{\mathbb{N}}$ with $x \neq y$

$\implies \mathrm{rng}(k)$ is uncountable ↯

WIP: More applications of the Gödel problem

Theorem

Consider problems

$$f, g :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$$

such that it is decidable if for a $\sigma \in \mathbb{N}^*$ there exists an extension $x \in \mathbb{N}^{\mathbb{N}}$ of σ with $x \in dom(f)$.

Then

$$f * G \leq_W g * G$$

entails

$$f \leq_W g$$

Corollary

We have

$$G^{[n+1]} \not\leq_W G^{[n]}$$

for any $n \in \mathbb{N}$.

Finite computable transformations

Computable realizer

A partial, multivalued $f :\subseteq \mathcal{D}^n \rightrightarrows \mathcal{D}$ is computable if there is a code $e \in \mathbb{N}$ such that

$$\begin{array}{ccc} \subseteq (\mathbb{N}^{\mathbb{N}})^n & \xrightarrow{\varphi_e} & \mathbb{N}^{\mathbb{N}} \\ \downarrow{\scriptstyle \mathsf{deg}^n} & & \downarrow{\scriptstyle \mathsf{deg}} \\ \subseteq \mathcal{D}^n & \xrightarrow{f} & \mathcal{D} \end{array}$$

for any $(\mathbf{a}_0, \dots, \mathbf{a}_{n-1}) \in \mathrm{dom}(f)$ and $x_0, \dots, x_{n-1} \in \mathbb{N}^{\mathbb{N}}$ with $\mathrm{deg}(x_i) = \mathbf{a}_i$ for all $i < n$:

The program $\varphi_e^{\langle x_0, \dots, x_{n-1} \rangle}$ terminates with a value $y \in \mathbb{N}^{\mathbb{N}}$ such that $\mathrm{deg}(y) \in f(\mathbf{a}_0, \dots, \mathbf{a}_{n-1})$ holds.

Computable realizer: Example

Example (Join)

$$f : \mathcal{D}^2 \to \mathcal{D} \quad \text{with} \quad (\mathbf{a}, \mathbf{b}) \mapsto \{\mathbf{a} \vee \mathbf{b}\}$$

is computable:

$$\varphi_e^{\langle x,y \rangle} = \langle x, y \rangle$$

Example (Jump inversion)

$$\mathrm{J}_{\mathcal{D}}^{-1} :\subseteq \mathcal{D} \rightrightarrows \mathcal{D} \quad \text{with} \quad \mathbf{a} \mapsto \{\mathbf{b} \mid \mathbf{b}' = \mathbf{a}\}$$

is *not* computable (Brattka, Hendtlass, Kreuzer 2017).

Characterization: finite version

Theorem

For any computable $f :\subseteq \mathcal{D}^n \rightrightarrows \mathcal{D}$, *there exists a set* $N \subseteq n$ *such that*

$$\bigvee_{i \in N} \mathbf{a}_i \in f(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1})$$

holds for all degrees $(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}) \in dom(f)$.

Example: jump inversion

Assume, for contradiction, that the jump inversion

$$\mathsf{J}_{\mathcal{D}}^{-1} :\subseteq \mathcal{D} \rightrightarrows \mathcal{D} \quad \text{with} \quad \mathbf{a} \mapsto \{\mathbf{b} \mid \mathbf{b}' = \mathbf{a}\}$$

is computable.

By our theorem, we have one of the following:

- $\mathbf{0} \in \mathsf{J}_{\mathcal{D}}^{-1}(\mathbf{a})$ for all $\mathbf{a} \in \mathcal{D}$ or
- $\mathbf{a} \in \mathsf{J}_{\mathcal{D}}^{-1}(\mathbf{a})$ for all $\mathbf{a} \in \mathcal{D}$.

Both lead to a contradiction.

(Already a “pointwise” disjunction would lead to a contradiction.)

Example: compositional product of PA (1/2)

Assume, for contradiction, that the reduction

$$f * f \leq_W f$$

holds for a problem $f :\subseteq \mathcal{D} \rightrightarrows \mathcal{D}$ such that

- $\mathbf{0} \in \mathrm{dom}(f)$ (can be replaced with "$\mathrm{dom}(f) \neq \emptyset$"),
- $\mathrm{rng}(f) \subseteq \mathrm{dom}(f)$,
- no solution of $f(\mathbf{a})$ is computable from $\mathbf{a}$, for any $\mathbf{a} \in \mathcal{D}$.

Then, we have

$$g \leq_W f * f \quad \text{and hence} \quad g \leq_W f$$

for

$$g : \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathcal{D}^2 \quad \text{with} \quad g(\cdot) := \{(\mathbf{a}, \mathbf{b}) \in \mathcal{D}^2 \mid \mathbf{0} < \mathbf{a} < \mathbf{b}\}$$

Example: compositional product of PA (2/2)

We have

$$g \leq_{\mathrm{W}} f$$

for a problem $f :\subseteq \mathcal{D} \rightrightarrows \mathcal{D}$ and

$$g : \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathcal{D}^2 \quad \text{with} \quad g(\cdot) := \{(\mathbf{a}, \mathbf{b}) \in \mathcal{D}^2 \mid \mathbf{0} < \mathbf{a} < \mathbf{b}\}$$

realized by $h :\subseteq \mathcal{D}^2 \rightrightarrows \mathcal{D}^2$ and $k : \mathbb{N}^{\mathbb{N}} \to \mathcal{D}$.

Thus, we have one of

- $\mathbf{0} \in (\pi_0 \circ h)(\mathbf{0}, \mathbf{a})$ for all $\mathbf{a} \in f(\mathbf{0})$, or
- $\mathbf{a} \in (\pi_0 \circ h)(\mathbf{0}, \mathbf{a})$ for all $\mathbf{a} \in f(\mathbf{0})$.

Contradiction by case distinction $\quad \Longrightarrow \quad f * f <_{\mathrm{W}} f$

This result is applicable to NON, PA, ...

Characterization: finite version with multiple outputs

Theorem

For any computable $f :\subseteq \mathcal{D}^n \rightrightarrows \mathcal{D}^m$, there exist sets $N_0, \ldots, N_{m-1} \subseteq n$ such that

$$\left(\bigvee_{i \in N_0} \mathbf{a}_i, \ldots, \bigvee_{i \in N_{m-1}} \mathbf{a}_i \right) \in f(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1})$$

holds for all degrees $(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}) \in dom(f)$.

Example: compositional product of PA (2/2, again)

We have

$$g \leq_{\mathrm{W}} f$$

for a problem $f :\subseteq \mathcal{D} \rightrightarrows \mathcal{D}$ and

$$g : \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathcal{D}^2 \quad \text{with} \quad g(\cdot) := \{(\mathbf{a}, \mathbf{b}) \in \mathcal{D}^2 \mid \mathbf{0} < \mathbf{a} < \mathbf{b}\}$$

Thus, we have a computable $h :\subseteq \mathcal{D}^2 \rightrightarrows \mathcal{D}^2$ with

$$\{(\mathbf{0}, \mathbf{0}), (\mathbf{0}, \mathbf{a}), (\mathbf{a}, \mathbf{0}), (\mathbf{a}, \mathbf{a})\} \cap h(\mathbf{0}, \mathbf{a}) \neq \emptyset.$$

for all $\mathbf{a} \in f(\mathbf{0})$.

$\Longrightarrow \quad f * f <_{\mathrm{W}} f$

This result is applicable to NON, PA, ...

Proof: splitting

Splitting

For all $i < n$ and

$s_0, \ldots, s_{n-1} \in 2^*$ together with $s'_i \in 2^*$, there are

$t_0, \ldots, t_{n-1} \in 2^*$ together with $t'_i \in 2^*$ such that φ_e^σ and φ_e^τ with

- $\sigma := \langle s_0 * t_0, \ldots, s_{i-1} * t_{i-1}, \boldsymbol{s_i * t_i}, s_{i+1} * t_{i+1}, \ldots, s_{n-1} * t_{n-1} \rangle$
- $\tau := \langle s_0 * t_0, \ldots, s_{i-1} * t_{i-1}, \boldsymbol{s'_i * t'_i}, s_{i+1} * t_{i+1}, \ldots, s_{n-1} * t_{n-1} \rangle$

are incomparable.

Proof: first case

Proposition

If "Splitting" holds:

Then, for any degrees $\mathbf{a}_0, \ldots, \mathbf{a}_{n-1} \in \mathcal{D}$*, there are sequences* $x_0, \ldots, x_{n-1} \in \mathbb{N}^{\mathbb{N}}$ *with* $\mathbf{a}_i = \deg(x_i)$ *for all* $i < n$*, such that*

$$\bigvee_{i<n} \mathbf{a}_i = \deg(f(\langle x_0, \ldots, x_{n-1} \rangle))$$

holds.

Proof: uniform splitting

Splitting

For all $i < n$ and
$s_0, \ldots, s_{n-1} \in 2^*$ together with $s_i' \in 2^*$, there are
$t_0, \ldots, t_{n-1} \in 2^*$ together with $t_i' \in 2^*$ such that φ_e^σ and φ_e^τ with

- $\sigma := \langle s_0 * t_0, \ldots, s_{i-1} * t_{i-1}, \boldsymbol{s_i * t_i}, s_{i+1} * t_{i+1}, \ldots, s_{n-1} * t_{n-1} \rangle$
- $\tau := \langle s_0 * t_0, \ldots, s_{i-1} * t_{i-1}, \boldsymbol{s_i' * t_i'}, s_{i+1} * t_{i+1}, \ldots, s_{n-1} * t_{n-1} \rangle$

are incomparable.

$$\Downarrow$$

Uniform splitting

For any $k \in \mathbb{N}$, there are sequences
$t_0^0, \ldots, t_{n-1}^0, t_0^1, \ldots, t_{n-1}^1 \in 2^*$ s.t. for all sequences
$s_0, \ldots, s_{n-1} \in 2^k$ and distinct $p, q \in 2^n$,
φ_e^σ and φ_e^τ are incomparable for

- $\sigma := \left\langle s_0 * t_0^{\boldsymbol{p}(0)}, \ldots, s_{n-1} * t_{n-1}^{\boldsymbol{p}(n-1)} \right\rangle$ and
- $\tau := \left\langle s_0 * t_0^{\boldsymbol{q}(0)}, \ldots, s_{n-1} * t_{n-1}^{\boldsymbol{q}(n-1)} \right\rangle$.

Proof: first case

Uniform splitting

For any $k \in \mathbb{N}$, there are sequences $t_0^0, \ldots, t_{n-1}^0, t_0^1, \ldots, t_{n-1}^1 \in 2^*$ s.t. for all sequences $s_0, \ldots, s_{n-1} \in 2^k$ and distinct $p, q \in 2^n$, φ_e^σ and φ_e^τ are incomparable for

- $\sigma := \langle s_0 * t_0^{\boldsymbol{p}(0)}, \ldots, s_{n-1} * t_{n-1}^{\boldsymbol{p}(n-1)} \rangle$ and
- $\tau := \langle s_0 * t_0^{\boldsymbol{q}(0)}, \ldots, s_{n-1} * t_{n-1}^{\boldsymbol{q}(n-1)} \rangle$.

$$\Downarrow$$

Define computable $g_i : 2^\mathbb{N} \to 2^\mathbb{N}$ for $i < n$.

E.g.: $g_i(0101\ldots) := t_{i,0}^0 * t_{i,1}^1 * t_{i,2}^0 * t_{i,3}^1 * \ldots$

- $\deg(x) = (\deg \circ g_i)(x)$ for all $x \in 2^\mathbb{N}$ and $i < n$,
- $\bigvee_{i<n} \deg(x_i) = \deg(f(\langle g_0(x_0), g_1(x_1), \ldots \rangle))$
  for any $x_0, \ldots, x_{n-1} \in 2^\mathbb{N}$

Proof: second case

If "Splitting" does not hold at $i = n-1$:

There are $s_0, \ldots, s_{n-1} \in 2^*$ together with $s'_{n-1} \in 2^*$ such that for all $t_0, \ldots, t_{n-1} \in 2^*$ with $t'_{n-1} \in 2^*$, the sequences φ_e^σ, φ_e^τ for

- $\sigma := \langle s_0 * t_0, \ldots, s_{n-2} * t_{n-2}, \boldsymbol{s_{n-1}} * \boldsymbol{t_{n-1}} \rangle$
- $\tau := \langle s_0 * t_0, \ldots, s_{n-2} * t_{n-2}, \boldsymbol{s'_{n-1}} * \boldsymbol{t'_{n-1}} \rangle$

are comparable.

Define $g :\subseteq (\mathbb{N}^\mathbb{N})^{n-1} \to \mathbb{N}^\mathbb{N}$ such that φ_e^ρ is an infinite sequence for

$$\rho := \langle s_0 * x_0, \ldots, s_{n-2} * x_{n-2}, s_{n-1} * g(x_0, \ldots, x_{n-2}) \rangle$$

and all $x_0, \ldots, x_{n-1} \in 2^\mathbb{N}$ with $(\deg(x_0), \ldots, \deg(x_{n-1})) \in \operatorname{dom}(f)$.

Proof: second case

Define $g :\subseteq (\mathbb{N}^{\mathbb{N}})^{n-1} \to \mathbb{N}^{\mathbb{N}}$ such that φ_e^{ρ} is an infinite sequence for

$$\rho := \langle s_0 * x_0, \dots, s_{n-2} * x_{n-2}, s_{n-1} * g(x_0, \dots, x_{n-2}) \rangle$$

and all $x_0, \dots, x_{n-1} \in 2^{\mathbb{N}}$ with $(\deg(x_0), \dots, \deg(x_{n-1})) \in \mathrm{dom}(f)$.

Define $f' :\subseteq \mathcal{D}^{n-1} \rightrightarrows \mathcal{D}$ with

$$\mathrm{dom}(f') := \{(\mathbf{a}_0, \dots, \mathbf{a}_{n-2}) \mid (\mathbf{a}_0, \dots, \mathbf{a}_{n-2}, \mathbf{a}_{n-1}) \in \mathrm{dom}(f) \text{ for some } \mathbf{a}_{n-1} \in \mathcal{D}\}$$

$$f'(\langle \mathbf{a}_0, \dots, \mathbf{a}_{n-2} \rangle) := \bigcap \{f(x) \mid x := (\mathbf{a}_0, \dots, \mathbf{a}_{n-2}, \mathbf{a}_{n-1}) \in \mathrm{dom}(f)\}$$

This function is computable witnessed by φ_e, $s_0, \dots, s_{n-1}$ and g.

$\Rightarrow$ Apply induction hypothesis

Proof: second case

Induction hypothesis yields $N \subseteq n-1$ such that

$$\bigvee_{i \in N} \mathbf{a}_i \in f'(\mathbf{a}_0, \ldots, \mathbf{a}_{n-2})$$

for all $(\mathbf{a}_0, \ldots, \mathbf{a}_{n-2}) \in \mathrm{dom}(f')$.

Hence,

$$\bigvee_{i \in N} \mathbf{a}_i \in f(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1})$$

for all $(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}) \in \mathrm{dom}(f)$.

Infinite computable transformations

Characterization: infinite version

Theorem

For any computable $f :\subseteq \mathcal{D}^{\mathbb{N}} \rightrightarrows \mathcal{D}$, one of the following two holds:

a) *There is a finite set $N \subseteq \mathbb{N}$ such that*

$$\bigvee_{n \in N} \mathbf{a}_n \in f((\mathbf{a}_n)_{n \in \mathbb{N}})$$

holds for any sequence of degrees $(\mathbf{a}_n)_{n \in \mathbb{N}} \in dom(f)$.

b) *We have*

$$\deg((x_n)_{n \in \mathbb{N}}) \in f((\deg(x_n))_{n \in \mathbb{N}})$$

for any coded sequence $(x_n)_{n \in \mathbb{N}} \subseteq \mathbb{N}^{\mathbb{N}}$ of number sequences with $(\deg(x_n))_{n \in \mathbb{N}} \in dom(f)$.

In particular, $f((\deg(x_n))_{n \in \mathbb{N}})$ contains any Turing degree above $\deg((x_n)_{n \in \mathbb{N}})$.

WIP: Applications to linear orders (1/2)

Definition

Let X be a linear order. The order $\overline{X}$ results from X if we add top and bottom elements should they not exist.

Corollary

For any linear order X defined on the natural numbers, let $w_X : \mathcal{D} \rightrightarrows \mathcal{D}^{\mathbb{N}}$ be the Weihrauch problem defined by

$$w_X(\mathbf{a}) := \{(\mathbf{b}_n)_{n\in\mathbb{N}} \mid \mathbf{a} < \mathbf{b}_n \text{ for all } n \in \mathbb{N} \\ \text{and } \mathbf{b}_n < \mathbf{b}_m \text{ if and only if } n <_X m \text{ for all } n, m \in \mathbb{N}\}.$$

For any two linear orders X and Y, if $w_X \leq_W w_Y$ holds, then there is an embedding of X into $\overline{Y}$.

WIP: Applications to linear orders (2/2)

Definition

We consider

$$\mathrm{INC}, \mathrm{DEC} :\subseteq \mathcal{D} \rightrightarrows \mathcal{D}^{\mathbb{N}}$$

with

$$\mathrm{INC}(\mathbf{a}) := \{(\mathbf{b}_i)_{i\in\mathbb{N}} \mid \mathbf{a} < \mathbf{b}_i < \mathbf{b}_j \text{ for all } i < j \in \mathbb{N}\} = w_\omega$$
$$\mathrm{DEC}(\mathbf{a}) := \{(\mathbf{b}_i)_{i\in\mathbb{N}} \mid \mathbf{a} < \mathbf{b}_j < \mathbf{b}_i \text{ for all } i < j \in \mathbb{N}\} = w_{-\omega}$$

Corollary

The Weihrauch problems INC and DEC are incomparable.

Corollary

We have the strict Weihrauch reduction

$$\widehat{NON^{\diamond}} <_W NON^{\infty}$$