CCA 2026

Twenty-Third International Conference

on

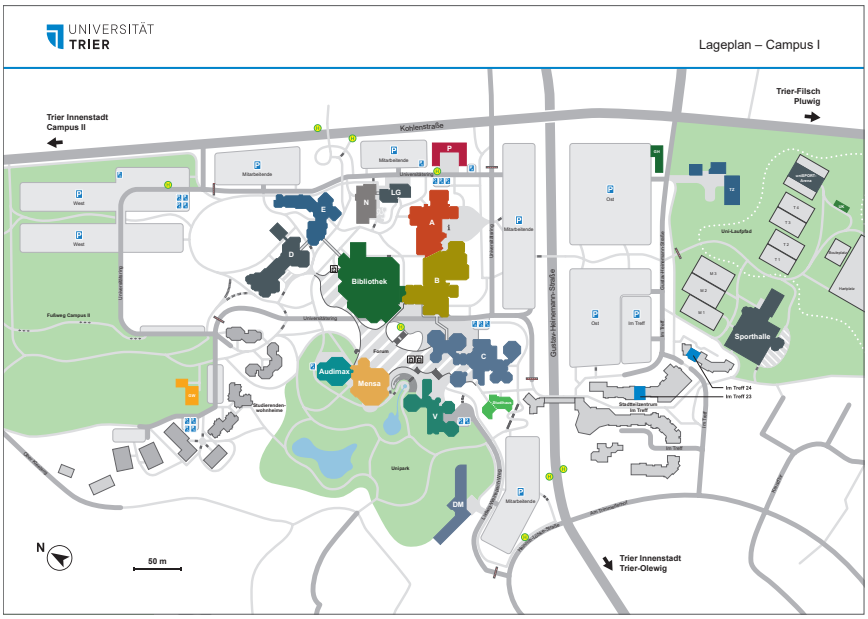
Computability and Complexity in Analysis

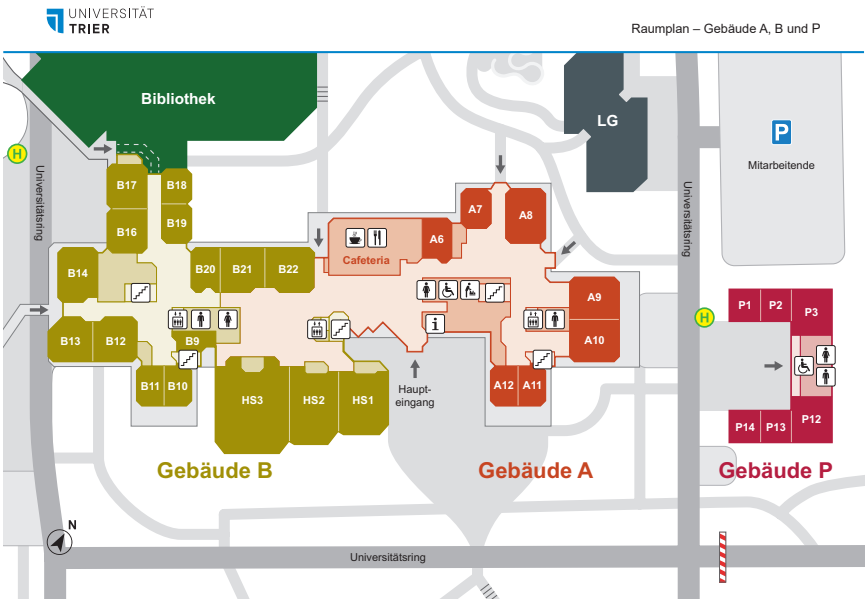
July 29-31, 2026, Trier, Germany

Abstracts

1 Conference Venue

Maps of campus and building A/B with lecture halls HS 1 and HS 3:





2 Schedule

Wednesday, July 29, 2026	
08:00 - 08:45	Registration in front of HS 3
08:45 - 09:00	Welcome Address, Organizatorial Information, HS 3
Session 1.1	HS 3, chair: Vasco Brattka
09:00 - 11:00	**Olivier Bournez** Ordinary Differential Equations as a Universal Language for Computability and Complexity: From Polynomial Time to the Hyperarithmetical Hierarchy (**Tutorial**)
11:00 - 11:30	Coffee Break
Session 1.2	HS 1, chair: Peter Hertling
11:30 - 12:00	**Schröder, Matthias** A Computable Version of the Tychonoff Theorem
12:00 - 12:30	**Iljazović, Zvonko & Validžić, Lucija & Vasung, Patrik** Computable sets and symmetry points
12:30 - 13:00	**Aguilar, Miguel & Aguilera, Juan & Fernandez-Duque, David** The Reverse Mathematics of the Mountain Pass Theorem
13:00 - 14:00	Lunch Break
14:00	Start of the Excursion
18:00	Conference Dinner in the City Center

Thursday, July 30, 2026

Session 2.1	HS 1, chair: Zvonko Iljazović
09:00 - 10:00	**Park, Sewon** Transforming classical mathematics in type theory into oracle computation (**Invited Talk**)
10:00 - 10:30	**Hertling, Peter** On the Complexity of the Saddle Transition Problem for Hyperbolic Toral Automorphisms
10:30 - 11:00	**Brattka, Vasco & Sorg, Christopher** Computability of the Hahn–Banach Theorem Revisited
11:00 - 11:30	Coffee Break
Session 2.2	HS 1, chair: Holger Thies
11:30 - 12:30	**Blanc, Manon** Robustness in Dynamical Systems: when Allowing Noise Makes the Undecidable Effectively Decidable (**Invited Talk**)
12:30 - 13:00	**Čelar, Matea & Iljazović, Zvonko & Jelić, Matea & Tarandek, David** Effective decompositions of continua with embedded arcs
13:00 - 14:00	Lunch Break
Session 2.3	HS 1, chair: Matthias Schröder
14:00 - 15:00	**de Brecht, Matthew** Effective subcategories of quasi-Polish spaces (**Invited Talk**)
15:00 - 15:30	**Brattka, Vasco & Chirache, Guillaume** Uniform Computability of PAC Learning
15:30 - 16:00	Coffee Break
Session 2.4	HS 1, chair: Sewon Park
16:00 - 17:00	**Pradic, Cécilia** The Myhill isomorphism theorem does not generalize much (in a sense) (**Invited Talk**)
17:00 - 17:30	**Burnik, Konrad** On Computability of Universal Block-Diagonalization via Finite Group Representations

Friday, July 31, 2026

Session 3.1	HS 1, chair: Eike Neumann
09:00 - 10:00	**Dinowitz, Emma** Point-to-set principles in dynamical systems (**Invited Talk**)
10:00 - 10:30	**Burnik, Konrad & Iljazović, Zvonko** Computable Homeomorphisms from One-Point Metric Bases
10:30 - 11:00	Coffee Break
Session 3.2	HS 1, chair: Cécilia Pradic
11:00 - 11:30	**Neumann, Eike & Tembo, Margret** Robust Safety and Persistence Verification for Discrete-TimeLinear Systems
11:30 - 12:00	**Sanders, Sam** A Notion of Reducibility for Third-Order Arithmetic
12:00 - 13:00	Lunch Break
Session 3.3	HS 1, chair: Sam Sanders
13:00 - 14:00	**Nandakumar, Satyadev** Effective Banach-Mazur games and an application to the Poincaré recurrence theorem for category (**Invited Talk**)
14:30 - 14:30	**Nandakumar, Satyadev & Pulari, Subin & S, Akhil** Constructive Dimension via Φ-Coverings and Faithfulness of Cantor Coverings
14:30 - 15:00	Coffee Break
Session 3.4	HS 1, chair: Vasco Brattka
15:00 - 15:30	**Wyeth, Cole & Hutter, Marcus** The Arithmetical Hierarchy of Real Functions
15:30 - 16:00	**Adejoh, Rosemary & Jakoby, Andreas & Mohanty, Sneha & Schindelhauer, Christian** Complexity Bounds for Illumination in 2D using Reflections
16:00 - 16:30	**Titov, Ivan** Speedability as a self-translation property
16:30 - 16:45	Closing Address

3 Invited Talks

Robustness in Dynamical Systems: when Allowing Noise Makes the Undecidable Effectively Decidable

Manon Blanc

IT-Universitet i København

First, what we call here a (general) dynamical system is a tuple $\mathcal{H} = (X, f)$ with X a set and $f : X \to X$. It is a mathematical model usually used in applied science to describe a system evolving over time. For example, in ecology and evolutionary biology, the evolution of a population placed in some environment with antagonistic survival strategies is usually represented by dynamical systems. In computer science, a Turing machine is also a dynamical system. The evolutions themselves are modelled with differential equations, ordinary or partial. We only study the ODE framework here for discrete-time (described by discrete ODEs) and continuous-time dynamical systems. The main issue here is that reasoning about systems evolving on the real numbers leads to undecidability, even for rather elementary properties of dynamical systems. It comes from the fact that Turing machines can be embedded into dynamical systems (even very simple ones). We focus on the reachability problem:

> **Reach**:
> Input: A dynamical system $\mathcal{H} = (X, f)$, two points $x, y \in X$
> Output: Is there a finite trajectory between x and y in $\mathcal{H}$?

In the general case, **Reach** is not decidable, even under reasonable hypotheses, such as having a computable dynamic function. In this case, it is just computably enumerable. However, various results in the literature have shown that decision procedures exist when one restricts oneself to robust systems, with a suitably chosen notion of robustness. More formally, here, a discrete-time dynamical system $\mathcal{H} = (X, f)$ is robust if and only if for any trajectory $(x_t)_{t\in\mathbb{N}} \in X^{\mathbb{N}}$, for any $\epsilon > 0$, and for any $t \in \mathbb{N}$, $d(x_{t+1}, f(x_t)) < \epsilon$. This definition can be generalised to continuous-time systems. In particular, in the field of verification, if reachability is insensitive to small perturbations, then decision procedures for reachability exist. In the context of logical theories over real numbers, it has been established that decision procedures exist by focusing on properties insensitive to arbitrarily small perturbations. Also, in general, once we have decidability, the complexity of deciding the procedures tends to explode. It is therefore not effectively feasible, even in the case of distributed systems.

We proposed a unified theory explaining these statements, which were established in different contexts, within a uniform framework.

More fundamentally, although all these statements are only about computability issues, we also consider complexity theory aspects. *We prove that robustness to a certain precision is intrinsically linked to the complexity of the decision procedure.* When a system is robust, it is logical to quantify its level of perturbation. We prove that assuming robustness to a polynomial perturbation, namely an allowed error of $2^{-p(n)}$ with p a polynomial and n an integer, on the precision leads to a characterisation of **PSPACE**. In other words, **we prove that the precision of the computations is intrinsically linked to the spatial complexity, while the length or time of the trajectories is intrinsically linked to the temporal complexity**. The systems studied are general and not necessarily distributed. It resulted in a publication in CSL 2024 [1], and more recently in Journal of Complexity [2].

Now, in verification, reachability is not the only fundamental property that is undecidable in the general case. Safety, invariance, viability, inevitability and bisimulation are also standard questions of verification and control theory. They are each a least or a greatest fixed point of a monotone operator built from the dynamics. We connect the previous results with several neighbouring theories, by treating every monotone fixed point at once, and we separate two layers. The first is topological and uses no computability: robustness is the coincidence of the exact object with the limit of its vanishing perturbations, which reads as the pseudo-orbit tracing, or shadowing, property. It identifies the perturbed object with Conley's chain-reachable set, so that the finite graph computing it is the box graph of computational Conley theory and its Morse decomposition converges to that of the system. The second is effective and adds a computable presentation: the coincidence becomes an algorithm and a finite Knaster–Tarski certificate. Decidability is uniform across the class, and the precision it costs is the metric entropy of the state space, giving polynomial space for a single fixed point and exponential space under alternation, with δ-decidability recovered as a special case. The contribution is more about this bridge than any single theorem: a duality of least and greatest fixed points under which perturbation, shadowing, chain recurrence, descriptive complexity and δ-decision appear as facets of one phenomenon, needing neither a Lipschitz constant nor any finiteness assumption on the dynamics.

References

[1] Manon Blanc and Olivier Bournez. Quantifying the robustness of dynamical systems. relating time and space to length and precision. In Aniello Murano and Alexandra Silva, editors, *32nd EACSL Annual Conference on Computer Science Logic, CSL 2024, February 19-23, 2024, Naples, Italy*, volume 288 of *LIPIcs*, pages 17:1–17:20, Naples, Italy, February 2024. Schloss Dagstuhl - Leibniz-Zentrum für Informatik. doi:10.4230/LIPIcs.CSL.2024.17.

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Effective subcategories of quasi-Polish spaces

Matthew de Brecht (Kyoto, Japan)

In previous work, we constructed the category of quasi-Polish spaces as a represented space, and showed that limits, coproducts, and standard powerspace monads are computable in the sense of Type Two Theory of Effectivity (TTE).

In this talk, we show that some important sub-categories of quasi-Polish spaces (in particular the subcategories of overt discrete quasi-Polish spaces and compact Hausdorff quasi-Polish spaces) can be constructed as effective quasi-Polish spaces (i.e., effective internal categories of the category of quasi-Polish spaces).

To show that the constructions are natural, we show that Stone duality (for both objects and morphisms) is computable, in the sense that the dual contravariant functors and the natural transformations demonstrating their adjointness are computable.

Point-to-set principles in dynamical systems

Emma Dinowitz (New York, USA)

The point-to-set principle for Hausdorff and packing dimensions is a framework for studying the dimensions of a set using a pointwise algorithmic information quantity called the effective dimension. In dynamical systems, Hausdorff and packing dimensions are often studied in terms of their relation to the dynamical dimension quantities of entropy, pressure, and Barreira–Schmeling (BS) dimension. We prove point-to-set principles for these quantities and introduce a method of proving relationships between types of dimension quantities of sets by proving pointwise relationships for an upper set of oracles. We apply this to generalize some classical theorems in the dimension theory of dynamical systems as well as resolve several conjectures on the Hausdorff/packing dimension of level sets of Lyapunov exponents for geodesic flows on surfaces of nonpositive curvature.

Effective Banach-Mazur games and an application to the Poincaré recurrence theorem for category

Satyadev Nandakumar

Abstract

The classical Banach-Mazur game characterizes sets of first category in a topological space. In this work, we show that an effectivized version of the game yields a characterization of sets of effective first category. Using this, we provide a game-theoretic proofs of effective versions of some results in topology. First, we discuss the effective category of the set of Liouville numbers. Further, we show an effective theorem in dynamical systems, namely the category version of Poincaré Recurrence. The Poincaré Recurrence Theorem for category states that for a homeomorphism without open wandering sets, the set of non recurrent points forms a first category (meager) set. As an application of the effectivization of the Banach-Mazur game, we show that such a result holds true in effective settings as well.

Transforming classical mathematics in type theory into oracle computation*

Sewon Park

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Keywords: Oracle computation in type theory, proof transformation, program extraction

In the practice of formalizing classical mathematics, classical principles such as the axiom of choice or the law of excluded middle are typically assumed as global functions realizing them. For instance, the Mathlib library of the Lean theorem prover assumes a single function that, given any nonempty type, chooses an element of it.

This way of assuming classical principles, however, makes classical results hard to reuse in a constructive (or computational) setting, where it is inconsistent to assume a functional realizer that works uniformly for all inputs. Instead, we consider oracle computation as a controlled and consistent way of assuming classical principles inside a constructive type theory: rather than having a function, we restrict the classical principles so that they may be accessed only by querying an oracle. That is, for a type X, we consider instead an inductive type OX whose constructors either embed an element of X or build a term of OX from an answer to a query, for example, an element of a nonempty type. This way, classical results can even be refined into computations whenever the oracle queries can be resolved.

In this talk, I present progress from our ongoing project with Andrej Bauer (University of Ljubljana, Slovenia) and Danel Ahman (University of Tartu, Estonia) on transforming classical results in type theory into oracle computations. Previous work, including [AB26], treated the case where the classical principle is propositional, that is, where the type of answers to each oracle query is a subsingleton, which includes the law of excluded middle. However, it did not extend well to the nonpropositional case, where a single query may have more than one possible answer, as in Lean's formulation of the axiom of choice. The focus of our work is to extend the theory to the nonpropositional case by lifting the oracles into a forcing setting where the nonpropositional oracle becomes propositional. We also present a prototype Lean metaprogram implementing the transformation on a small fragment of Lean's type theory.

References

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The Myhill isomorphism theorem does not generalize much (in a sense)

Cécilia Pradic (Swansea, UK)

The Myhill isomorphism theorem is a computable version of Cantor-Bernstein for subsets of $\mathbb{N}$. It states that, for any two subsets $A, B \subseteq \mathbb{N}$ that are inter-reducible via computable injections, there is a computable bijection $\mathbb{N} \to \mathbb{N}$ that preserves them. Myhill's proof consists of a back-and-forth, which in fact applies more broadly: the statement remains true in intuitionistic logic when the word "computable" is dropped throughout. The original theorem can be recovered by interpreting the constructive proof in Kleene's realizability.

The version of theorem without computability assumptions remains true in classical logic if $\mathbb{N}$ is replaced by an arbitrary set X, but this is not the case constructively. Bauer asked if there is a nice class of sets X for which it does hold constructively. After checking there is no hope for this class of sets to be closed under basic operations like disjoint unions, we will see that a version of this generalized Myhill isomorphism theorem holds for the conatural numbers $\mathbb{N}_\infty$ by adapting the usual back-and-forth construction and assuming Markov's principle. However, this does not extend too much: this fails for $2 \times \mathbb{N}_\infty$, $\mathbb{N} + \mathbb{N}_\infty$ as well as Cantor space $2^{\mathbb{N}}$. We are going to see why those failures are of different flavours, and sketch how to make this more precise by using oracle modalities.

I will then sketch some directions for possible further work on problems of the same flavour, such as set division and refinement of Cantor-Bernstein. I will explain how I think oracle modalities in the style of Weihrauch reducibility might be an interesting tool to study such non-constructive theorems that involves higher types, and how oracle modalities may allow in a sense to refine arguments from Weihrauch reducibility in a purely logical setting.

4 Tutorial

Ordinary Differential Equations as a Universal Language for Computability and Complexity: From Polynomial Time to the Hyperarithmetical Hierarchy

Bournez, Olivier (Paris, FR)

This tutorial argues that ordinary differential equations form a universal language for computability and complexity. Tuning a small number of parameters, the regularity of the right-hand side, the length of the solution, the precision admitted on the input, and the continuous or discrete nature of the iteration, recovers a wide range of complexity classes within a single algebraic setting, from FAC^0 up to FPTIME, FPSPACE, and the hyperarithmetical hierarchy. Models historically treated as separate, including the General Purpose Analog Computer, recurrent neural networks under the deep-narrow regime, and chemical reaction networks under mass-action kinetics, appear as complementary readings of the same object. A central result is a polynomial-time four-way equivalence between Turing time, polynomial-ODE solution length, fixed-width neural network depth, and Boolean circuit size.

The tutorial will be accompanied by live demonstrations on analog computing hardware brought to Trier.

5 Contributed Talks

Complexity Bounds for Illumination in 2D using Reflections

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We revisit the computational complexity of the Illumination Problem in various settings and contribute some new insights building on the seminal work of Aronov et al. [1]. Given a light source, we inquire whether a specific point is illuminated. We prove that this problem is $\mathcal{NP}$-hard for two dimensions, where mirrors are represented as line segments. Meanwhile, for the upper bound, we show that Illumination in k-dimensions, with mirrors represented as k-simplexes with rational point coordinates and linearly bounded reflections, can be solved in $\mathcal{NP}$. In the illumination problem one assumes that light follows a straight line and is reflected according to the physical law of reflection on mirrored surfaces that are placed in k-dimensional space. The physical law of reflection states that the angle of incidence equals the angle of reflection for the light ray hitting the surface of a mirror. We only consider plane objects as mirrors, i.e. line segments in 2D and polygons embedded in a plane for three dimensions (as well as a generalization for higher dimensions). While Penrose et al. [2] introduced the illumination problem only for enclosed regions, we do not restrict mirrors to enclosing the light source.

For the illumination problem, Penrose et al. [2] proved that there exists a closed room with curved (elliptic and straight) surface, where a light source cannot illuminate all points within. For polygonal rooms Tokarsky [3] proved that there is a 27 sided polygonal where some internal points cannot be illuminated by a light source. A smaller construction of a 24 sided polygon was later provided by Castro [4].

The analysis of the complexity of the illumination problem has been studied as a special form of a *visibility problem*, see the chapter edited by Toth et al. in [5]. For a single reflection Aronov et al. [6] showed that this region has a combinatorial complexity of $\Theta(n^2)$, and can be computed by a $\mathcal{O}(n^2 \log^2 n)$ time bounded algorithm. For k reflections the combinatorial complexity grows exponential with $\Omega((n/k) - \mathcal{O}(1))^{2k})$ for an n-sided polygon. They also provide an $\mathcal{O}(n^{2k} \log n)$-time bounded algorithm to compute all these paths in space $\mathcal{O}(n^{2k})$.

We show that one-way (semi-transparent) mirrors increase the complexity and in order to understand this in more detail, we distinguish their behavior of splitting and merging beams, see Table I. So, on one side of a splitting one-way mirror, light rays pass through and produce a reflection, but on the opposite side, it is a perfect mirror. A merging mirror is fully transparent on one side and on the other side fully reflective, while a normal one-way-mirror is semi-transparent for both sides and produces reflections for both directions, see Figure 1. Note that for the ray tracing problem we have used the same set of mirrors and showed new results for this problem in [8].

Illumination complexity	Dimensions	Type of mirrors	Number of reflections
$\mathcal{NP}$	1	a, b, c, d	polynomial
$\mathcal{NP}$-Hard	2	a	exponential
$\mathcal{NP}$-Hard	2	a + b	linear

TABLE I
SUMMARY OF OUR CONTRIBUTIONS. THE SYMBOLS '+' REFER TO 'AND' AND ',' FOR 'OR'. THE TYPES OF MIRRORS ARE DEPICTED IN FIGURE 1

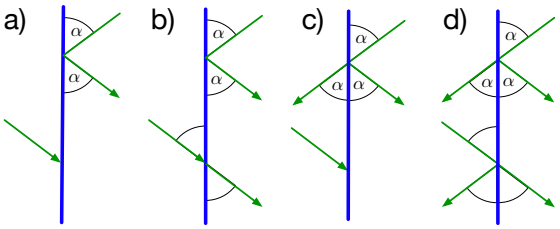

Fig. 1. a) full, b) one-way merging, c) one-way splitting, and d) one-way mirror.

We consider the *Illumination* problem as the question, whether for a given set of mirrors from a source point a ray can be emitted that reaches a given target point, while passing or being reflected by the given set of mirrors. We may limit the number t of reflections the rays needs to reach the target for this problem and call this variant t-bounded Illumination.

Furthermore, we assume that the coordinates of all points are given as rational numbers, for which the binary representation is part of the input.

Our first results shows an upper bound that for any Euclidean space with $k \geq 2$ dimensions, where the coordinates of the light source and the target point, as well as the mirrors are given as rational numbers.

Theorem 1. *The k-dimensional illumination problem with rational input values and a polynomial number of reflections is in $\mathcal{NP}$ if we allow simple plain mirrors, plain merging, plain splitting and one-way mirrors.*

The proof of this theorem as well as all other necessary details can be found in our technical report [9]. Our lower bound contributions hold already in two-dimensional space.

Theorem 2. *The illumination problem restricted to simple plane mirrors in 2D is $\mathcal{NP}$-hard, if an exponential number of reflections is allowed.*

We prove this by a reduction from the subset sum problem to the Illumination problem. Recall that, in the subset sum problem, a set of numbers $x_1, \ldots, x_n$ is given, and the task is to determine whether a given value s is the sum of a subset of these values. In our technical report [9], we give an explicit construction in which a light source sends a narrow light cone into a corridor, where tilted wall segments represent the numbers $x_1, \ldots, x_n$ by their tilting angles. This is done in such an intricate way that possible beams can either be reflected by the tilted segments once or pass to the neighboring straight wall. In this way, the light beams acquire all combinations of tilt angles, which are then tested at a target corridor representing the sum s. Thus, the illumination problem solves the subset sum problem.

It turns out that the number of reflections in this proof grows exponentially with the number of tilted segments. However, the use of merging mirrors can reduce this number significantly.

Theorem 3. *The illumination problem restricted to simple plane mirrors and merging mirrors in 2D is $\mathcal{NP}$-hard, even if only a linear number of reflections is allowed.*

As shown in [9] in great detail, we use the reduction from subset sum in the proof of Theorem 2. The reason for the exponential number of reflections is that the light cone narrows with every new tilted segment. Thus, many reflections are required to ensure that light rays corresponding to all combinations of tilted segments intersect. However, if we use parallel merging mirrors placed at distances that are powers of two, this merging of light cones can be achieved with significantly fewer reflections.

Note that in this case the lower bound and the upper bound from Theorem 1 match. Thus, the illumination problem with merging and simple plane mirrors is the first illumination problem known to be $\mathcal{NP}$-complete.

Still, the question whether the Illumination problem is $\mathcal{NP}$-hard for a polynomial number of reflections with simple mirrors remains open.

References

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The Reverse Mathematics of the Mountain Pass Theorem

Miguel Aguilar, Juan Aguilera, David Fernandez-Duque

We investigate the metamathematical aspects of the Mountain Pass Theorem (MPT) of Ambrosetti and Rabinowitz. MPT is a minimax theorem establishing sufficient conditions for differentiable functionals to possess critical points and has a wide catalog of applications in PDEs, variational methods, calculus on manifolds, and other domains. Unlike other results of its kind, the critical points provided by MPT are typically saddle points.

We prove that MPT for finite dimensional Hilbert spaces is equivalent to Weak König's Lemma (WKL_0) over the axiomatic system RCA_0 of computable mathematics. Therefore, MPT is not computably valid.

Our proof of MPT deviates from usual proofs and requires establishing computable existence theorems for ODEs on separable Banach spaces, as well as a New Deformation Lemma of independent interest. Doing so isolates precisely where non-computability assumptions are needed and shows that MPT for separable spaces can be proved computably if the derivative of the functional I in question is assumed to possess Lipschitz constants on bounded sets.

UNIFORM COMPUTABILITY OF PAC LEARNING

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In recent years, computability properties of PAC learning (which stands for "probably approximately correct" learning) have found increasing interest in the research community and several authors have investigated computable variants of PAC learning. In most of these cases, PAC learning is looked at from a non-uniform perspective and the focus is on computability properties of individual concept classes and their learners. We take a different standpoint and investigate PAC learning from a uniform perspective [1], using the tools of computable analysis and Weihrauch complexity. This uniform approach allows us to study both the non-constructiveness of the Fundamental Theorem of Statistical Learning and the computability-theoretic complexity of the problem: *given* a concept class of finite VC dimension, *find* a learner for it.

Definition 1 (PAC learning)**.** Let $\varnothing \neq \mathcal{C} \subseteq \mathcal{H} \subseteq 2^{\mathbb{N}}$ be hypothesis classes, let $A : \mathcal{S} \to \mathcal{H}$ be a learner and let $m : \mathbb{N} \to \mathbb{N}$ be a function that we call *sample complexity*. We say that A *PAC learns* $\mathcal{C}$ *relative to* $\mathcal{H}$ *with* m if for all $i, j \in \mathbb{N}$ and $n \geq m\langle i, j\rangle$ and for every probability distribution $\mathcal{D}$ over $\mathbb{N} \times \{0, 1\}$, $\mathbb{P}_{S\sim\mathcal{D}^n}\left[L_{\mathcal{D}}(A(S)) \leq \inf_{h\in\mathcal{C}} L_{\mathcal{D}}(h) + 2^{-i}\right] > 1 - 2^{-j}$ holds.

If $\mathcal{C} = \mathcal{H}$, then we say that A *(properly) PAC learns* $\mathcal{C}$ *with* m. If $\mathcal{C} \subseteq \mathcal{H}$ and $\mathcal{H} = 2^{\mathbb{N}}$, then we say that A *improperly PAC learns* $\mathcal{C}$ *with* m.

Theorem 2 (Fundamental Theorem of Statistical Learning)**.** Let $\varnothing \neq \mathcal{C} \subseteq \mathcal{H} \subseteq 2^{\mathbb{N}}$. Then

$$\mathcal{C} \text{ is PAC learnable relative to } \mathcal{H} \iff \mathrm{VCdim}(\mathcal{C}) \text{ is finite.}$$

For the most interesting direction "$\Longleftarrow$" we essentially obtain the following $\forall\exists$–statement for the case of proper PAC learning:

$$(\forall \mathcal{C})(\mathrm{VCdim}(\mathcal{C}) < \infty \Longrightarrow (\exists (A, m) \in \mathcal{L} \times \mathbb{N}^{\mathbb{N}})(A \text{ PAC learns } \mathcal{C} \text{ with } m)).$$

In order to analyze the Weihrauch complexity of this statement, we interpret the corresponding computational problem as a multivalued map that takes $\mathcal{C}$ as input and produces a possible pair (A, m) as output. We have made the choice to restrict our study to topologically closed concept classes $\mathcal{C}$, which is reasonable since the VC dimensions of a set and its closure are identical. The class of closed sets has continuum cardinality and can be dealt with using standard techniques of computability theory and computable analysis.

Altogether, we have studied three different general versions of PAC learning. Besides the usual concepts of proper and improper PAC learning, it turned out to be useful to consider also a variant of PAC learning that we refer to as *relative PAC learning*, where besides the concept class $\mathcal{C}$, we also provide the larger class $\mathcal{H} \supseteq \mathcal{C}$ of hypotheses that are permitted as results. In this case, we consider the class $\mathcal{C}$ given by positive information and the permitted hypotheses are described negatively, as this allows one to exclude unwanted hypotheses.

Among other results, we prove that proper PAC learning from positive information is equivalent to the limit operation on Baire space, whereas improper PAC learning from positive information is closely related to Weak Kőnig's Lemma and even equivalent to it, when we have some negative information about the admissible hypotheses. If arbitrary hypotheses are allowed, then improper PAC learning from positive information is still in a finitary DNC range, which implies that it is non-deterministically computable, but does not allow for probabilistic algorithms. Finally, we also classify the complexity of the VC dimension operation itself, which is a problem that is of independent interest.

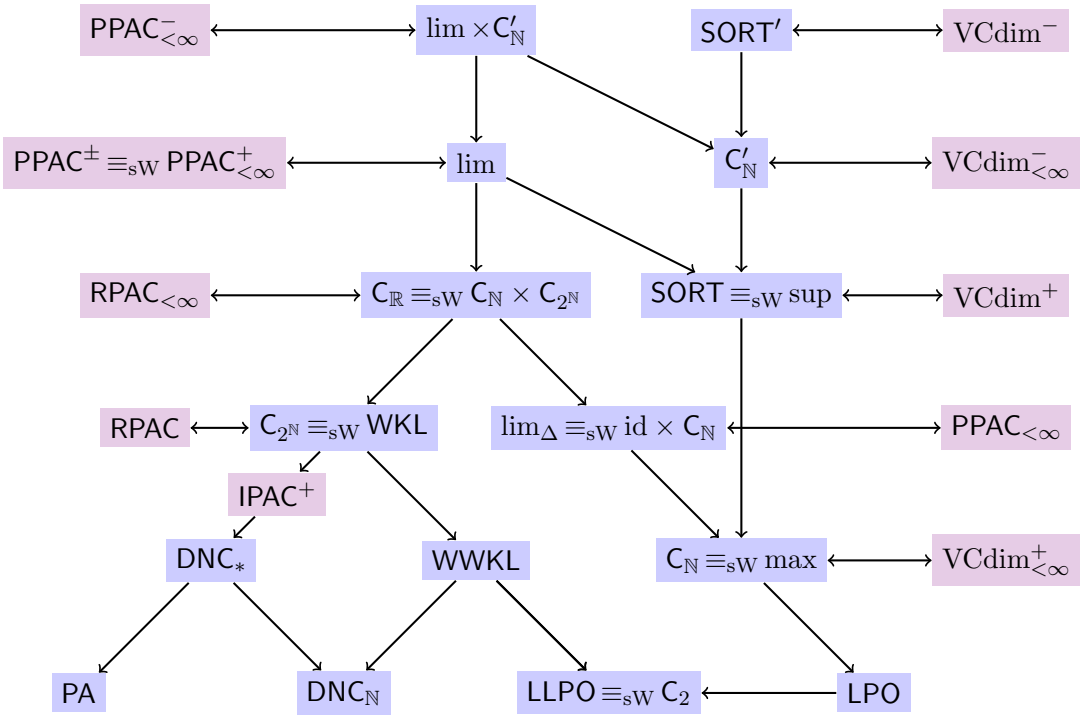

Figure 1: PAC learning in the strong Weihrauch lattice with several benchmark problems. All relevant definitions, precise statements and proofs are included in [1].

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Computability of the Hahn–Banach Theorem Revisited

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Motivation and problem. The classical Hahn-Banach theorem states that if X is a normed space, $A \subseteq X$ is a linear subspace, and $f\colon A \to \mathbb{R}$ is bounded and linear, then f extends to a bounded linear functional $g\colon X \to \mathbb{R}$ with $\|g\| = \|f\|$. In computable analysis the natural question is of uniform matter: given effective descriptions of X, of a closed linear subspace $A \subseteq X$, and of f, how much non-computability is needed to produce such an extension? Previous work showed that, when the ambient Banach space is allowed to depend on the input, the Hahn-Banach problem is Weihrauch equivalent to weak *König*'s lemma WKL [4]. The main point of the present work is that this variability is not needed: the fixed Banach space ℓ^1 already realizes the full degree [1].

For a computable normed space X, we therefore study the multivalued problem HBT_X: the input is a non-zero bounded linear functional $f\colon A \to \mathbb{R}$ on a closed linear subspace $A \subseteq X$ given by an effectively enumerable dense sequence, and the output is any linear extension $g\colon X \to \mathbb{R}$ with $\|g\| \leq \|f\|$. Since $g|_A = f$ already implies $\|g\| \geq \|f\|$, the inequality actually forces equality. We also consider the one-step problem HBT^1_X, where one only has to extend f from A to $A + \mathbb{R}x$ for a given point $x \in X$. We compare such problems by Weihrauch reducibility: $P \leq_{\mathrm{W}} Q$ means that one can solve P using one oracle call to Q together with computable pre- and post-processing [3].

Main results. Our results give three exact classifications:

$$\mathsf{HBT}_{\ell^1_2} \equiv_{\mathrm{W}} \mathsf{LLPO}, \qquad \mathsf{HBT}^1_{\ell^1} \equiv_{\mathrm{W}} \mathsf{IVT}, \qquad \mathsf{HBT}_{\ell^1} \equiv_{\mathrm{W}} \mathsf{WKL}.$$

Here $\ell^1_2 = (\mathbb{R}^2, \|\cdot\|_1)$, IVT is the intermediate value theorem, and LLPO is the lesser limited principle of omniscience. Moreover, the last equivalence remains true even if the input subspace is given by its distance function

$$d_A(x) := \inf_{y \in A} \|x - y\|,$$

that is, even for located closed subspaces of ℓ^1 [1]. Since LLPO, IVT, and WKL form three distinct benchmark degrees, these results isolate three genuinely different computational regimes of the Hahn-Banach extension problem.

This fixed-space phenomenon is far from automatic. If every bounded linear functional on every subspace of X has a unique norm-preserving extension, then HBT_X becomes single-valued; every single-valued problem below WKL with values in a computable metric space is computable. In particular, if the dual space X' is strictly convex, then HBT_X is computable [1]. Hence the sequence spaces ℓ^p with $1 < p < \infty$ are computationally tame in this sense, while $(\ell^1)' = \ell^\infty$ is not strictly convex. This makes ℓ^1 a valid canonical space on which one should look for full uniform non-computability, and our theorem shows that this intuition is exactly right.

Proof ideas and red line. The proof has a clear conceptual progression from local extension to global extension. The classical one-step Hahn-Banach theorem says that an extension $g\colon A + \mathbb{R}x \to \mathbb{R}$ with $\|g\| \leq 1$ exists exactly when

$$\sup_{y \in A}\big(f(y) - \|x - y\|\big) \leq g(x) \leq \inf_{y \in A}\big(f(y) + \|x - y\|\big).$$

Thus the computational content of one step is: choose any real number between a lower semi-computable supremum and an upper semi-computable infimum. This is precisely an IVT-type task, so $\mathsf{HBT}^1_X \leq_{\mathrm{W}} \mathsf{IVT}$ for every computable normed space X [1]. Repeating the one-step construction along a fundamental sequence of X gives a new proof of the general upper bound $\mathsf{HBT}_X \leq_{\mathrm{W}} \mathsf{WKL}$; in Weihrauch terms this iteration is captured by the infinite-loop operator and mirrors the role of dependent choice in the classical proof [1, 2].

For the converse $\mathsf{IVT} \leq_{\mathrm{W}} \mathsf{HBT}^1_{\ell^1}$, we encode a nested interval problem

$$a_0 \leq a_1 \leq \cdots \leq b_1 \leq b_0$$

into a subspace of $\ell^1(\mathbb{N} \times \{0,1\})$. Put

$$\alpha_n := \frac{a_n - b_n}{2} < 0, \qquad \beta_n := \frac{a_n + b_n}{2},$$

define $u_n := e_{n,0} + \alpha_n e_{n,1}$ and $v_n := e_{n,0} - e_{n+1,0}$, let A be the closure of $\operatorname{span}\{u_n, v_n : n \in \mathbb{N}\}$, and define f by $f(v_n) = 0$ and $f(u_n) = \beta_n$. For

$$x := \sum_{n=0}^{\infty} 2^{-n-1} e_{n,0},$$

every norm-preserving extension to $A + \mathbb{R}x$ determines a real y with $y \in [a_n, b_n]$ for every n, hence $y \in \bigcap_n [a_n, b_n]$. Therefore one-step Hahn-Banach on ℓ^1 has exactly the strength of the intermediate value theorem [1].

To obtain $\mathsf{WKL} \leq_{\mathrm{W}} \mathsf{HBT}_{\ell^1}$, we start from the standard separation problem: given maps $p, q \colon \mathbb{N} \to \mathbb{N}$ with disjoint ranges, compute a set $B \subseteq \mathbb{N}$ such that

$$\operatorname{range}(p) \subseteq B \subseteq \mathbb{N} \setminus \operatorname{range}(q).$$

This problem is Weihrauch equivalent to WKL [4, 3]. Gherardi and Marcone encode each such instance into a Banach space $X_{p,q}$, a closed subspace $A_{p,q} \subseteq X_{p,q}$, and a norm-one functional $f_{p,q} \colon A_{p,q} \to \mathbb{R}$ whose norm-preserving extensions compute a separator [4]. Our key new step is to eliminate the variable ambient space: we construct, uniformly in (p, q), a computable linear isometry

$$F_{p,q} \colon X_{p,q} \to \ell^1.$$

The construction is blockwise. Each two-dimensional block norm appearing in $X_{p,q}$ is first transformed into an ℓ_2^∞-norm, and a fixed linear isometry $S \colon \ell_2^\infty \to \ell_2^1$ then transfers it into ℓ^1. Transporting $f_{p,q}$ and $A_{p,q}$ through $F_{p,q}$ moves the entire coding into the single fixed space ℓ^1, which proves $\mathsf{HBT}_{\ell^1} \equiv_{\mathrm{W}} \mathsf{WKL}$ [1].

The two-dimensional result isolates the first non-trivial branching phenomenon. In ℓ_2^1, ambiguity appears exactly when the one-dimensional input subspace passes through a corner of the diamond-shaped unit ball; deciding which supporting side carries an extension is exactly an LLPO-type choice. Conversely, Ishihara's idea extracts the sign information of a real parameter from any norm-preserving extension on $\operatorname{span}\{(1, r)\}$, giving $\mathsf{HBT}_{\ell_2^1} \equiv_{\mathrm{W}} \mathsf{LLPO}$ [1, 5]. Finally, because computable linear isometries preserve distance-function representations of closed subspaces, the fixed-space lower bound transfers to located subspaces as well [1].

Contribution of the talk. The talk will emphasize this red line: local binary ambiguity in dimension two, interval choice in the one-step problem, and full tree-like non-determinism in the complete extension problem. In this way the Hahn-Banach theorem becomes a clean case study in how fine geometric features of a Banach space are reflected exactly by Weihrauch degrees.

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On Computability of Universal Block-Diagonalization via Finite Group Representations

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We introduce the notion of a *universal block-diagonalizer* for a class of matrices and study its computability in the Type-2 model of real computation [6]. A matrix $U \in O(n)$ is a *universal block-diagonalizer* for a class of matrices $\mathcal{A} \subseteq M_n(\mathbb{R})$ if $U^T A U$ is block-diagonal for every $A \in \mathcal{A}$, and U does not depend on the choice of A.

When $\mathcal{A}$ is a subalgebra, the Wedderburn–Artin theorem guarantees that such a U exists if and only if $\mathcal{A}$ is semisimple.

A subalgebra $\mathcal{A} \subseteq M_n(\mathbb{R})$ is *semisimple* if it is isomorphic to a direct sum of matrix algebras over division rings. Over $\mathbb{R}$, the only finite-dimensional division rings are $\mathbb{R}$, $\mathbb{C}$, and the quaternions $\mathbb{H}$ (Frobenius). In that case, there exists $U \in O(n)$ such that for every $A \in \mathcal{A}$,

$$U^T A U = \operatorname{diag}(A_1, \ldots, A_k),$$

where $A_i \in M_{s_i}(\mathbb{R})$ with $s_1 + \cdots + s_k = n$. Each block A_i is constrained by the corresponding division ring D_i: if $D_i = \mathbb{R}$, then A_i is an arbitrary $r_i \times r_i$ real matrix ($s_i = r_i$); if $D_i = \mathbb{C}$, then $s_i = 2r_i$ and A_i is a block matrix of 2×2 blocks of the form $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$; if $D_i = \mathbb{H}$, then $s_i = 4r_i$ and A_i has an analogous structure with 4×4 blocks representing quaternion entries.

In the Type-2 model of computability, a real number is specified by a *Cauchy name* for a real number x: a sequence $(q_m)_{m \in \mathbb{N}}$ of rationals satisfying $|q_m - x| \leq 2^{-m}$ for all $m \in \mathbb{N}$, and a *Type-2 machine* is a Turing machine that reads and writes such sequences on infinite tapes. A real matrix is specified by Cauchy names for each of its entries.

A semisimple subalgebra generated by a matrix whose entries are given by Cauchy names need not have a computable universal block-diagonalizer.

Proposition 1 ([7])**.** *For every $n \geq 2$, there is no Type-2 machine that, given Cauchy names for the n^2 entries of a symmetric matrix $A \in M_n(\mathbb{R})$, computes Cauchy names for the entries of an orthogonal eigenvector matrix of A.*

For $n = 2$, consider the family from Example 18 of [7]: for $\epsilon \in \mathbb{R}$, $\epsilon \neq 0$, let

$$B(\epsilon) = e^{-1/\epsilon^2} \begin{pmatrix} \cos(2/\epsilon) & \sin(2/\epsilon) \\ \sin(2/\epsilon) & -\cos(2/\epsilon) \end{pmatrix}, \qquad B(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This is a computable one-parameter family of symmetric matrices. When $\epsilon \neq 0$, the matrix $B(\epsilon)$ has two distinct eigenvalues $\pm e^{-1/\epsilon^2}$; when $\epsilon = 0$, it has a single eigenvalue 0 of multiplicity 2. Suppose for contradiction that a Type-2 machine computes an orthogonal eigenvector matrix of $B(\epsilon)$ uniformly in ϵ. But the unit eigenvectors of $B(\epsilon)$ are $\pm(\cos(1/\epsilon), \sin(1/\epsilon))$ and $\pm(-\sin(1/\epsilon), \cos(1/\epsilon))$, whose directions fail to converge as $\epsilon \to 0$. Hence, no eigenvector matrix can depend continuously on ϵ. Further, since every computable map is continuous, none can depend computably on ϵ either. Contradiction. For $n > 2$, embed $B(\epsilon)$ in the upper-left 2×2 block of the $n \times n$ identity matrix and apply an analogous argument.

Since an orthogonal eigenvector matrix of a symmetric matrix A simultaneously diagonalizes every matrix that is a polynomial in A, it is in particular a universal block-diagonalizer for any semisimple subalgebra containing A. This shows that universal block-diagonalizers are not uniformly Type-2 computable for general semisimple subalgebras.

The obstacle to computing a universal block-diagonalizer is that the algorithm must know the sizes $s_1, \ldots, s_k$ of the diagonal blocks, and these sizes are not Type-2 computable from continuous data alone (since rank over $\mathbb{R}$ is not computable [7, Theorem 7]). When the algebra arises as the commutant of a finite group representation, however, the block sizes are determined by *integer* multiplicities of irreducible representations. These integers can be expressed as inner products of characters (finite sums of traces of representation matrices) and are therefore approximable from the given data. Since they are known a priori to be integers, approximating to precision less than $1/2$ and rounding recovers them exactly. This transforms the non-computable rank problem into a computable rounding problem.

The representation-theoretic definitions needed to make this precise are recalled below (see [2, 3]).

A *representation* of a finite group G is a group homomorphism $\rho\colon G \to \mathrm{GL}_n(\mathbb{R})$, where n is called the *degree* of ρ. A subspace $W \subseteq \mathbb{R}^n$ is *ρ-invariant* if $\rho(g)W \subseteq W$ for all $g \in G$. The representation ρ is *irreducible* if $\mathbb{R}^n$ has no ρ-invariant subspace other than $\{0\}$ and $\mathbb{R}^n$ itself. The *character* of ρ is the function $\chi_\rho\colon G \to \mathbb{C}$ defined by $\chi_\rho(g) = \mathrm{tr}(\rho(g))$.

By *Maschke's theorem*, every representation of a finite group over $\mathbb{R}$ decomposes as a direct sum of irreducible subrepresentations. Let $\rho_1, \ldots, \rho_k$ denote the pairwise non-isomorphic irreducible representations of G over $\mathbb{R}$, with characters $\chi_1, \ldots, \chi_k$. The irreducible characters are exactly computable from the Cayley table of G. Every representation ρ decomposes as

$$\rho \cong m_1\rho_1 \oplus \cdots \oplus m_k\rho_k,$$

where $m_i \in \mathbb{N}$ is the *multiplicity* of ρ_i in ρ. The *i-th isotypic subspace* of ρ is $V_i = \mathrm{im}(P_i)$, where

$$P_i = \frac{n_i}{|G|} \sum_{g \in G} \chi_i(g)\, \rho(g)$$

is the orthogonal projection onto V_i; it has dimension $d_i = m_i \cdot n_i$, where n_i is the degree of ρ_i.

The *commutant* of ρ is the subalgebra

$$\mathcal{C}(\rho, G) = \{A \in M_n(\mathbb{R}) \mid A\rho(g) = \rho(g)A \text{ for all } g \in G\}.$$

Many structured matrix classes in numerical linear algebra (circulant, centrosymmetric, bisymmetric, and block-circulant matrices [1, 4]) arise as commutants of finite group representations.

The commutant $\mathcal{C}(\rho, G)$ is semisimple by Maschke's theorem and the Wedderburn–Artin theorem [2, 3]. For instance, circulant matrices form the commutant of the cyclic permutation representation of $\mathbb{Z}/\ell\mathbb{Z}$, and the discrete Fourier transform matrix (the classical universal block-diagonalizer in this case) has entries built from the irreducible characters of the group. Similarly, centrosymmetric matrices form the commutant of the exchange matrix representation of $\mathbb{Z}/2\mathbb{Z}$ [5]. For matrices commuting with a dihedral group D_m acting on $\mathbb{R}^n$, Theorem 2 yields a computable block-diagonalizer whose block structure reflects the one- and two-dimensional irreducible representations of D_m.

A universal block-diagonalizer is assembled from orthonormal bases of the isotypic subspaces $V_1, \ldots, V_k$. As noted above, computing an orthonormal basis for V_i requires knowing $\dim(V_i) = \mathrm{rank}(P_i)$. The finite group structure resolves this. The multiplicity m_i is given by

$$m_i = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g)\, \overline{\chi_i(g)},$$

and is a non-negative integer, being the number of copies of ρ_i in the decomposition of ρ. The trace $\chi_\rho(g) = \mathrm{tr}(\rho(g))$ is approximable from the Cauchy names for $\rho(g)$, while the irreducible characters $\chi_i(g)$ are exactly computable from the Cayley table. Since $m_i \in \mathbb{N}$, it can be exactly recovered by rounding its Type-2 approximation once the error is less than $1/2$. This yields the exact isotypic dimensions $d_i = m_i \cdot n_i$. With these dimensions obtained, Theorem 11 of [7] provides Type-2 computable orthonormal bases for each isotypic subspace. The columns of U realize the isotypic decomposition.

Theorem 2. *There exists a Type-2 machine $\mathcal{M}$ with the following property. Let G be a finite group of order ℓ and let $\rho\colon G \to \mathrm{GL}_n(\mathbb{R})$ be a representation. Suppose the input to $\mathcal{M}$ consists of:*

(i) $\ell, n \in \mathbb{N}$,

(ii) the $\ell \times \ell$ Cayley table of G encoded as a finite string over $\{1, \ldots, \ell\}$, and

(iii) for each $g \in G$, Cauchy names for the n^2 entries of $\rho(g)$.

Then $\mathcal{M}$ computes Cauchy names for the entries of an orthogonal matrix $U \in O(n)$ such that $U^T A\, U$ is block-diagonal with blocks of sizes $d_1, \ldots, d_k$ for every $A \in \mathcal{C}(\rho, G)$, where $d_i = m_i \cdot n_i$ is the dimension of the i-th isotypic subspace.

As an application of Theorem 2, we consider the problem of QR decomposition for matrices from $\mathcal{C}(\rho, G)$.

Corollary 3. *There exists a Type-2 machine $\mathcal{M}'$ with the following property. Suppose the input to $\mathcal{M}'$ consists of:*

(i) $n, k \in \mathbb{N}$ and the block sizes $d_1, \ldots, d_k \in \mathbb{N}$ with $d_1 + \cdots + d_k = n$,

(ii) Cauchy names for the n^2 entries of $U \in O(n)$ bringing $\mathcal{C}(\rho, G)$ into Wedderburn-adapted form (block-diagonal with each diagonal block a realification *of $M_{m_i}(D_i)$, i.e. each D_i-entry replaced by the real block representing it, as in the block forms above), and*

(iii) Cauchy names for the n^2 entries of a nonsingular $A \in \mathcal{C}(\rho, G)$.

Then $\mathcal{M}'$ computes Cauchy names for the entries of matrices $Q, R \in M_n(\mathbb{R})$ such that:

(a) $A = QR$,

(b) $Q \in \mathcal{C}(\rho, G)$ and $Q \in O(n)$,

(c) $R \in \mathcal{C}(\rho, G)$, and

(d) $U^T R U$ is upper triangular within each diagonal block of size d_i for $i \in \{1, \ldots, k\}$.

The integers k and $d_1, \ldots, d_k$ are all computable from the data (i)–(iii) of Theorem 2 by $\mathcal{M}$, which yields an isotypic-level block-diagonalizer. Computing a Wedderburn-adapted U from the same data is deferred to the full version of this work.

Taken together, Proposition 1 and Theorem 2 show that finite group symmetry resolves the non-computability of universal block-diagonalization (for the case of matrix class arising from a commutant) by providing the block dimensions as recoverable discrete data. This is an instance of *discrete advice* in the sense of Ziegler [8], where finite structural information renders an otherwise non-computable problem computable. Whether the finite group structure is the only such source of discrete advice remains open. Characterizing these sources is a natural question for further investigation.

In general, the connection between representation theory and computable analysis merits further investigation, as it suggests that algebraic structure may systematically inform the computability of numerical linear algebra problems.

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Computable Homeomorphisms from One-Point Metric Bases

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A compact subset S of Euclidean space $\mathbb{R}^n$ is said to be computable if S can be effectively approximated by a finite set of rational points with any given precision. If $f : [0,1] \to \mathbb{R}^n$ is a computable function, then $f([0,1])$ is a computable set. In particular, if $f : [0,1] \to \mathbb{R}^n$ is a computable injection, then $f([0,1])$ is a computable arc (i.e. a set which is computable and homeomorphic to $[0,1]$).

On the other hand, suppose $S \subseteq \mathbb{R}^n$ is a computable arc. Does there exist a computable injection $f : [0,1] \to \mathbb{R}^n$ such that $S = f([0,1])$? In other words, if S is a computable arc, is S computably homeomorphic to $[0,1]$? In general, the answer is negative, see Example 5.1 in [3].

We examine sufficient conditions under which the answer to the previous question is affirmative. Actually, we examine an equivalent formulation of the problem: if (X,d,α) is an effectively compact computable metric space such that (X,d) is homeomorphic to $[0,1]$, under what conditions (X,d,α) is computably homeomorphic to $[0,1]$?

A computable metric space (X,d,α) is effectively compact if (X,d) is compact and there is a computable function $\varphi : \mathbf{N} \to \mathbf{N}$ such that for each $k \in \mathbf{N}$

$$X = \bigcup_{i=0}^{\varphi(k)} B(\alpha_i, 2^{-k}).$$

Let (X,d) be a metric space. Let $S \subseteq X$, $S \neq \emptyset$. We say that S is a *resolving set* for (X,d) if

$$\forall x, y \in X \ (x \neq y \implies \exists s \in S, \ d(x,s) \neq d(y,s)).$$

The resolving set S of minimal cardinality is called a *metric basis* [2]. For one-point sets, the notions of resolving set and metric basis coincide. In this case, we identify the singleton with the point it contains.

We have the following result for arcs.

Theorem 1. *Let (X,d,α) be a computable metric space that is effectively compact and such that (X,d) is connected. Suppose (X,d) has a one-point metric basis. Then there exists a computable homeomorphism $f : [0,1] \to X$.*

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Furthermore, we have a similar result in a non-compact case. The effective compactness property is replaced with the effective covering property [1] and the requirement that (X, d) has compact closed balls.

Theorem 2. *Let (X, d, α) be a computable metric space with the effective covering property and compact closed balls. Suppose (X, d) is non-compact, connected and has a one-point metric basis. Then there exists a computable homeomorphism $f : [0, +\infty\rangle \to X$.*

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Effective decompositions of continua with embedded arcs*

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A continuum (compact and connected metric space) is said to be *decomposable* if it can be written as a union of two proper subcontinua. While it may seem that all nontrivial continua are decomposable, this is not the case; in fact, "most" continua are indecomposable [1]. Indecomposable continua naturally arise in various contexts in dynamical systems [2]. On the other hand, decomposability arises as a condition in the investigation of computable subcontinua of semicomputable chaniable continua [3].

In this talk, we consider continua endowed with the structure of a computable metric space and we investigate when a continuum is not merely decomposable, but expressible as a union of two *computable* proper subcontinua. We will call such continua *effectively decomposable*. Note that any effectively decomposable continuum is necessarily effectively compact. However, given an effectively compact continuum which is (topologically) decomposable, it is not obvious whether it is effectively decomposable, or in fact whether it contains a nontrivial computable proper subcontinuum.

Classically, a decomposable continuum can be characterised as a continuum which contains a proper subcontinuum with non-empty interior [4]. In the computable setting, an analogous proof gives the following result:

> *An effectively compact continuum which contains a computable proper subcontinuum whose interior is non-empty and c.e. open is effectively decomposable.*

We are interested in versions of the above result where effective properties of the subcontinuum are replaced by purely topological ones. Particularly, we focus on continua which contain a copy of the unit segment $[0,1]$ as a subcontinuum with non-empty interior (or, equivalenty, a copy of $\mathbb{R}$ as an open subset).

In [5], it was shown that any point in a semicomputable set which has an open neighbourhood homeomorphic to $\mathbb{R}$ also has a computable neighbourhood which is an arc with computable endpoints. Here, we build on this result to show that, whenever an effectively compact continuum contains (a homeomorphic copy of) $\mathbb{R}$ as an open subset, it must be effectively decomposable. More precisely, we prove the following:

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Theorem 1. *Let (X,d,α) be an effectively compact computable metric space. Suppose (X,d) is a continuum and there is an open subset of X homeomorphic to $\mathbb{R}$. Then X is effectively decomposable.*

Furthermore, we show that effectively compact, (classically) decomposable chainable continua are also effectively decomposable.

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On the Complexity of the Saddle Transition Problem for Hyperbolic Toral Automorphisms

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In a series of papers [2, 4, 3] Maller has studied the computational complexity of certain saddle transition problems for various kinds of diffeomorphisms of the two-dimensional torus in the framework of the Turing machine model [1]. In each case the result was that the respective problem is in Oracle NP for a highly restricted oracle. The main technical tool was in each case the shadowing lemma. In the first paper in this series, in [2], Maller considered hyperbolic toral automorphism on the two-dimensional torus. We shall reconsider this setting and will see that in this particular case one can prove a considerably stronger result.

The two-dimensional torus is $T := \mathbb{R}^2/\sim_{\mathbb{Z}^2}$ where the equivalence relation $\sim_{\mathbb{Z}^2}$ on $\mathbb{R}^2$ is defined by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \sim_{\mathbb{Z}^2} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} :\iff (x_1 - y_1 \in \mathbb{Z} \text{ and } x_2 - y_2 \in \mathbb{Z}).$$

Let us consider the set

$$H := \{A \mid A \text{ is a } 2 \times 2 \text{ matrix with entries in } \mathbb{Z}, \text{ with } \det(A) \in \{-1, 1\}, \text{ and with eigenvalues with absolute value} \neq 1\}.$$

It is easy to see that any matrix $A \in H$ naturally defines a homeomorphism f_A on T. These homeomorphisms are called *hyperbolic toral automorphisms*. It is well known that a point $x \in \mathbb{R}^2$ is a representant of a periodic point of f_A in T iff the coordinates of x are rational. Maller considered the following problem:

Given a matrix $A \in H$, a positive integer n, two different periodic points p and q of f_A, both of the same period n, a (small) positive rational number r and a positive integer N, does there exist a (rational) point $z \in B(p, r)$ such that $(f_A^n)^N(z) \in B(q, r)$?

Maller considered the case of symmetric matrices $A \in H$. We already mentioned that he showed that this problem is in Oracle NP for a very restricted oracle. Roughly, he showed that one can verify nonderministically

in polynomial time (with respect to the input data) whether N is close to the smallest integer χ such that there exists a (rational) point $z \in B(p,r)$ such that $(f_A^n)^\chi(z) \in B(q,r)$. It should be mentioned that, if instead of rational r one considers real r, then the function χ is discontinuous with respect to r, and therefore it is impossible to effectively determine the exact value of χ.

We are going to show that this problem can actually be solved in polynomial time (with respect to the input data) by a deterministic Turing machine: there is a deterministic Turing machine that computes in polynomial time a number $N \in \{\chi, \chi+1, \chi+2\}$. So, instead of a nondeterministic polynomial time algorithm we provide a deterministic polynomial time algorithm.

We should mention that the basic idea for the algorithm is already contained in the article [2] by Maller. The main technical tool employed by Maller is the shadowing lemma. But in the end of Section 3 of [2] he states a certain formula for χ which "depends entirely on global linearity" of (symmetric) hyperbolic toral automorphisms (cited from [2]). He does not make any use of this formula. He prefers to use the shadowing lemma, that is, "methods using pseudo-orbits" because they "can be generalized for functions on hyperbolic invariant sets" (cited from [2]). Our algorithm is essentially a careful and efficient implementation of a similar formula. Of course, we also need to make some additional considerations, in particular a careful estimation of the time needed by the algorithm.

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Computable sets and symmetry points

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A point $x \in \mathbf{R}^n$ is said to be computable if x can be effectively approximated by a rational point with any given precision. Similarly, a compact set $X \subseteq \mathbf{R}^n$ is said to be computable if X can be effectively approximated by a finite set of rational points with any given precision. Each nonempty computable set contains computable points, moreover they are dense in it.

Let $X \subseteq \mathbf{R}^n$. We say that $x \in X$ is a symmetry point of X if x is a fixed point for each isometry $f : X \to X$.

A nonempty computable subset X of $\mathbf{R}^n$ need not have a symmetry point (for example the unit circle in $\mathbf{R}^2$). However, if X is computable and has a symmetry point, does X have a computable symmetry point?

We show that the answer is negative in general. We also show (using results from [3]) that the set $s(X)$ of all symmetry points of X is semicomputable (which means that its complement $\mathbf{R}^n \setminus s(X)$ can be effectively exhausted by rational open balls).

If a nonempty compact set $X \subseteq \mathbf{R}^n$ is convex, then, by [2], there exists at least one symmetry point of X. We prove the following result.

Theorem 1. *Let $X \subseteq \mathbf{R}^n$ be a computable set. Suppose X is convex. Then the set $s(X)$ of all symmetry points of X is computable.*

In particular, a convex computable subset of $\mathbf{R}^n$ has a computable symmetry point.

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Constructive Dimension via Φ-Coverings and Faithfulness of Cantor Coverings

Abstract

Constructive dimension is a pointwise, effective analogue of Hausdorff dimension. Introduced by Lutz [2] via an effectivization of the notion of s-gales, it also admits a characterization (due to Mayordomo [4]) in terms of Kolmogorov complexity. The *point-to-set principle* by J.Lutz and N.Lutz [3] allows the Hausdorff dimension of a set to be expressed using the Kolmogorov complexities of its points, relative to a minimising oracle. Hausdorff dimension is defined using s-dimensional coverings of a set. In Euclidean space, constructive dimension is defined via dyadic coverings, or equivalently via binary representations of real numbers.

We study how these notions of dimension change when we restrict the class of coverings to a set Φ. Given a separable metric space and a countable dense family of coverings Φ, we define *Hausdorff* Φ*-dimension*, in which admissible covers are restricted to unions of elements from Φ. Fixing an effective enumeration δ of the elements of Φ, we further define *constructive* Φ*-dimension* of points. This effective notion satisfies a point-to-set principle: the Hausdorff Φ-dimension of any set can be characterized using the (relativized) Kolmogorov complexities of its points with respect to Φ. We also show that when the diameters of the Φ-coverings are computable relative to δ, the constructive Φ-dimension admits a direct Kolmogorov complexity characterization.

Finally, we apply these tools to study *faithfulness* of coverings Φ. A family of coverings Φ over Euclidean space is *faithful for constructive dimension* if the constructive Φ-dimension and the standard effective dimension agree at every point; faithfulness for Hausdorff dimension is defined analogously. It has been shown that coverings generated by continued fraction cylinders are not faithful for either dimension [6].

We focus on *Cantor coverings*, a natural generalization of base-b coverings. Given a sequence of integers $Q = (n_k)_{k\in\mathbb{N}}$ with $n_k > 1$, the Cantor covering over the unit interval at refinement level k partitions each interval into n_k equal subintervals. We characterize the faithfulness of such coverings in terms of the growth rate of the sequence: Cantor coverings are faithful for constructive dimension if and only if

$$\lim_{k\to\infty} \frac{\log n_k}{\log(n_1 \cdot n_2 \cdots n_{k-1})} = 0.$$

Using the point-to-set principle, we show that the same condition characterises Hausdorff dimension faithfulness of Cantor Coverings. This gives

an information theoretic proof of the result by Albeverio, Ivanenko, Lebid, and Torbin [1].

The arXiv version of the paper can be found here [5].

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Robust Safety and Persistence Verification for Discrete-Time Linear Systems

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We study safety and persistence verification for discrete-time linear systems

$$x_{k+1} = Ax_k, \qquad A \in \mathbb{R}^{d\times d},$$

with spectral radius bounded by 1 in the bit-model of real computation. We consider the problem of verifying whether a trajectory remains in a regular compact set forever, and the problem of verifying whether a trajectory eventually reaches a regular compact set and remains there forever. Such problems arise naturally in the verification of linear systems such as Markov chains [AAGT12], quantum automata [BJKP05] or control of marginally stable systems.

We further assume that we are given an *effective spectral gap* on the system under consideration: a rational number $\delta > 0$ such that every eigenvalue λ of A satisfies the implication $|\lambda| > 1 - \delta \implies |\lambda| = 1$. This allows us to effectively separate the strictly contracting spectrum from the unit-modulus spectrum.

More precisely, given a matrix $A \in \mathbb{R}^{d\times d}$ with $\rho(A) \leq 1$, an effective spectral gap $\delta > 0$, a regular compact set K, provided via two-sided distance information [Her05, Section 2.4], and an initial point $x \in \mathbb{R}^d$, we give complete procedures for verifying safety,

$$A^n x \in K \quad \text{for all } n \in \mathbb{N},$$

and persistence,

$$A^n x \in K \quad \text{for all sufficiently large } n.$$

We also treat the existential versions over a regular compact initial set I, asking whether some $x \in I$ is safe or persistent.

The safety verification problem was shown decidable in [NOW20] for matrices with exact algebraic entries and semialgebraic compact sets K represented using polynomials with exact algebraic coefficients. In this setting, it was shown complete for the $\exists\forall$-fragment of the theory of the reals in [DLN+21].

Our verification methods rely on rigorous numerical computation, avoiding symbolic algebraic methods. Thus, we expect our methods to be more computationally efficient in general than the algorithms proposed in [NOW20] and [DLN+21]. Further, since we work in the bit model, our procedures certify robustness: whenever a procedure halts, it has read only finite information from the input and hence certifies the property on a neighbourhood of the given instance. Accordingly, the relevant completeness notion is robust completeness: the procedures halt on all robust positive instances [Neu21, Section 2].

Our verification algorithms proceed as follows: Using the effective spectral gap, we compute the real spectral subspaces $E_{<1}(A)$ where A is contracting and $E_{=1}(A)$ where all eigenvalues of A lie on the unit circle, together with the corresponding spectral projections. We further compute a non-decreasing sequence of A-invariant subspaces

$$V_0 \subseteq V_1 \subseteq \cdots \subseteq \mathbb{R}^d$$

such that every unit-modulus eigenvalue of A_{V_n} is simple, and such that for some N,

$$V_N = E_{<1}(A) \oplus E_{=1}^{\text{simple}}(A),$$

where $E_{=1}^{\text{simple}}(A)$ is the space spanned by the eigenvectors of A for simple unit-modulus eigenvalues. If every unit-modulus eigenvalue of A is simple, then $V_N = \mathbb{R}^d$ for some N.

Given a V_n as above with $x \in V_n$, the ω-limit

$$\omega_A(x) = \left\{ y \in \mathbb{R}^d \mid \text{There exists an unbounded sequence } (k_n)_n \text{ with } \lim_{n\to\infty} A^{k_n} x = y \right\}$$

is a compact set. The idea behind verifying persistence is to verify if this compact set is contained in the interior of K. Unfortunately, the ω-limit does not depend continuously on A, so that this idea cannot be implemented directly. However, we show that we can compute a compact upper bound $\Omega(V_n, x, A) \supseteq \omega_A(x)$ to the ω-limit with the property that for all $\varepsilon > 0$ there exists a perturbation B of A with $\|B - A\| < \varepsilon$ such that

$$\omega_B(y) = \Omega(V_n, y, A) \qquad \text{for all } y \in V_n.$$

Thus $\Omega(V_n, x, A)$ agrees with the ω-limit set up to arbitrarily small perturbation. The bound $\Omega(V_n, x, A)$ is constructed by bringing the simple unit-modulus part into real block form, replacing $+1$-blocks by singletons, -1-blocks by two-point sets, and rotation blocks by circles. The Baire category theorem and the Kronecker approximation theorem yield that $\Omega(V_n, x, A)$ agrees with the ω-limit up to arbitrarily small perturbation.

For safety verification we proceed similarly, constructing a compact upper bound $O(V_n, x, A)$ of the full forward orbit with the property that for all $\varepsilon > 0$ there exists a perturbation B of A by at most ε such that the forward orbit of x under B is ε-close to $O(V_n, x, A)$ in the Hausdorff metric.

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A NOTION OF REDUCIBILITY FOR THIRD-ORDER ARITHMETIC

SAM SANDERS

Abstract. We introduce the concept of 'N-reduction', which is meant to play a role in higher-order arithmetic similar to that of Weihrauch reducibility in second-order arithmetic. In particular, N-reduction constitutes a relation 'is computationally stronger than' for third-order theorems *but* still based on Turing computability and Gödel's system T. Indeed, the notion of N-reduction is weaker than Kleene's S1-S9.

Short description

Turing's famous 'machine' model was the first intuitive framework for *computing with real numbers* ([17]). The latter is used to formulate *Weihrauch reducibility* ([1, 18]), a notion that allows for the comparison of the computational strength of theorems when the latter are formulated in the language of second-order arithmetic, i.e. via coding/representations of higher-order objects as reals.

It is a natural question whether one can formulate a similar framework for third-order arithmetic that makes no (or light) use of coding or representations. In answer to this question, we introduce *N-reduction*, which provides a relation 'is computationally stronger than' for *third-order* theorems of real analysis ([16]).

First of all, many theorems in e.g. real analysis can be given the form

$$(\forall Y : \mathbb{N}^{\mathbb{N}} \to \mathbb{N})(\exists x \in \mathbb{N}^{\mathbb{N}})A(Y, x). \tag{1}$$

A list of theorems that can be brought in the form (1) can be found in Example 1. We often use type theoretic notation, i.e. n^0 for type 0 objects $n \in \mathbb{N}$, x^1 for type 1 objects $x \in \mathbb{N}^{\mathbb{N}}$, and Y^2 for type 2 objects $Y : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$. Greek capitals $\Theta^3, \Lambda^3, \dots$ are for type 3 objects, which map type 2 objects to natural numbers.

Secondly, to compare the logical strength of theorems of the form (1), one establishes results of the following form over weak systems:

$$(\forall Y^2)(\exists x^1)A(Y, x) \to (\forall Z^2)(\exists y^1)B(Z, x). \tag{2}$$

The computational properties of (1) and (2) following Kleene's S1-S9 ([2]) are then be studied as follows: let Θ^3 and Λ^3 be *realisers* for the antecedent and consequent of (2) i.e. we have $(\forall Y^2)A(Y, \Theta(Y))$ and $(\forall Z^2)B(Z, \Lambda(Z))$.

A central computability theoretic question concerning (2) is whether a realiser Θ^3 for the antecedent of (2) computes, in the sense of S1-S9, a realiser Λ^3 for

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the consequent of (2), i.e. whether there is a Kleene algorithm with index $e \in \mathbb{N}$ satisfying the following:

$$(\forall \Theta^3)\big[(\forall Y^2)A(Y, \Theta(Y)) \to (\forall Z^2)B(Z, \{e\}(\Theta, Z))\big]. \tag{3}$$

Next, we list some theorems that have the form (1) and have been studied via the above paradigm based on (3) and Kleene's S1-S9. Many more theorems are equivalent to the theorems in the above list, as can be found in [11, 12].

Example 1 (Some representative theorems)**.**

- The Lindelöf, Heine-Borel, and Vitali covering theorems involving uncountable coverings ([3, 4, 6]),
- The Lebesgue number lemma ([5, 6]),
- The Baire category theorem ([5]),
- Convergence theorems for nets ([13–15]),
- Local-global principles like Pincherle's theorem ([6]),
- The uncountability of $\mathbb{R}$ and basic properties of countable sets ([8, 9]),
- Weak fragments of the Axiom of (countable) Choice ([7]).
- Basic properties of functions of bounded variation, like the Jordan decomposition theorem ([10]).

Thirdly, we now formulate a version of (3) based on Turing computability:

$$(\forall Z^2, x^1)\big[A(t(Z), x) \to [\{e\}^{s(Z,x)} \downarrow \wedge\, B(Z, \{e\}^{s(Z,x)})]\big], \tag{4}$$

where $s^{2\to 1}, t^{2\to 2}$ are terms of Gödel's T and '$\{e\}^x$' is the e-th Turing machine with oracle x^1. We note that (4) readily[1] implies (3). In line with the usual nomenclature, we call the antecedent and consequent of (2) 'problems' and say that

solving the problem $(\forall Z^2)(\exists y^1)B(Z, x)$ N***-reduces*** *to solving the problem* $(\forall Y^2)(\exists x^1)A(Y, x)$

in case (4) holds for the parameters mentioned. In case the term $s(Z, x)$ can be replaced by a term $u(x)$, i.e. no access to Z, we refer to (4) as **strong** N-reduction.

We shall discuss known results for N-reduction regarding the theorems in Example 1. A typical result on the uncountability of the reals ([16]) is as follows.

- $\mathsf{NIN}_{[0,1]}$: there is no injection from $[0, 1]$ to $\mathbb{N}$.
- **Cantor's theorem**: for a set $A \subset [0, 1]$ and $Y : [0, 1] \to \mathbb{N}$ injective on A, there is $x \in \big([0, 1] \setminus A\big)$.
- HBU: for any $\Psi : [0, 1] \to \mathbb{R}^+$, there are $x_0, \dots, x_k \in [0, 1]$ such that $\cup_{i \le k} B(x_i, \Psi(x_i))$ covers $[0, 1]$.

Theorem 2.

- *The problem* $\mathsf{NIN}_{[0,1]}$ *N-reduces to the Heine-Borel theorem* HBU*.*
- *Cantor's theorem N-reduces to the Heine-Borel theorem* HBU *restricted to Baire class* 2 *functions.*

We also discuss basic properties of nets and the operation that corresponds to the 'higher-order' Turing jump.

[1]For $e \in \mathbb{N}$ and $s^{2\to 1}, t^{2\to 2}$ as in (4), define $e_0 \in \mathbb{N}$ as the Kleene algorithm such that $\{e_0\}(\Theta, Z) := \{e\}^{s(Z, \Theta(t(Z)))}$, which is total by assumption.

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A Computable Version of the Tychonoff Theorem

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The Tychonoff Theorem states that the cartesian product of compact spaces is compact again. By the sequential Tychonoff Theorem[1], a countable product of compact subsets in a qcb-space X is compact in the product space $\mathsf{X}^{\mathbb{N}}$. It is an open question in Computable Analysis whether this theorem can be effectivised (cf. [2, Conjecture 3.29]).

Some partial results are already known. E.g., one can uniformly compute the product of two compact subsets of X and Y as a compact subset of $\mathsf{X} \times \mathsf{Y}$. Recently P. Uftring showed that the countable product $\mathsf{K}^{\mathbb{N}}$ of a co-c.e. compact space K with itself is co-c.e. compact again. However, it is unclear whether one can extend this to effectively forming the countable product of compact subsets K_i of a space X as a compact subset of the product space $\mathsf{X}^{\mathbb{N}}$.

The standard representation of compact subsets

The usual (multi-)representation $\delta_{\mathcal{K}(\mathsf{X})}$ of the set $\mathcal{K}(\mathsf{X})$ of compact subsets of a represented space X is based on the Sierpiński space $\mathbb{S}$. It embeds K into the hyperspace $\mathbb{S}^{\mathcal{O}(\mathsf{X})} \cong \mathbb{S}^{\mathbb{S}^{\mathsf{X}}}$ via the quantifier $\forall_K \colon \mathcal{O}(\mathsf{X}) \to \mathbb{S}$ which accepts an open subset U of X iff U contains all elements of K (cf. [1, 2, 4, 5, 6]). However, co-c.e. compactness of a space K means to be able to semi-decide whether an open subset V in the space K contains K. Unfortunately, in general it is not possible to computably convert an open set $V \in \mathcal{O}(\mathsf{K})$ into an open set $U \in \mathcal{O}(\mathsf{X})$ with $V = U \cap \mathsf{K}$. Indeed there is an admissibly represented qcb-space X which contains a $\delta_{\mathcal{K}(\mathsf{X})}$-computable sequence $(K_i)_i$ of compact subspaces which are not uniformly co-c.e. compact.

A new representation of compact subsets

We construct a new representation of compact subsets by replacing the Sierpiński space $\mathbb{S}$ by the Λ-space $\mathbb{L}$. The Λ-space $\mathbb{L}$ contains $\mathbb{S}$ as a subspace, has one additional point $\uparrow$ and belongs to the category EffLim of limit spaces equipped with an effectively admissible multirepresentation [5, 6]. A continuous function $v \colon \mathsf{X} \to \mathbb{L}$ can be viewed as a "partial open set", namely as the open subset $v^{-1}\{\top\}$ of the subspace $v^{-1}\{\bot, \top\}$. We embed subsets M of X into $\mathbb{L}^{\mathbb{L}^{\mathsf{X}}}$ via a modified quantifier $\forall_M \colon \mathbb{L}^{\mathsf{X}} \to \mathbb{L}$ which accepts a partial open set $v \colon \mathbb{L} \to \mathsf{X}$, if $M \subseteq v^{-1}\{\top\}$, rejects v, if there is some $x \in K$ with $v(x) = \bot$, and maps the other v's to the additional point $\uparrow$. It turns out that the map $\forall_M$ is a continuous function from $\mathbb{L}^{\mathsf{X}}$ to $\mathbb{L}$ if, and only if, M is a sequentially compact subset of X.

Similar to $\delta_{\mathcal{K}(\mathsf{X})}$, we define a multirepresentation $\delta_{\mathcal{SK}(\mathsf{X})}$ of the set $\mathcal{SK}(\mathsf{X})$ of all sequentially compact subsets of X by:

$$p \text{ is a name of } K \;:\Longleftrightarrow\; p \text{ is a name of } \forall_K \text{ w.r.t. the canonical multirepresentation for } \mathbb{L}^{\mathbb{L}^{\mathsf{X}}}$$

In the case that X is T_1 (i.e. all singletons are closed) $\delta_{\mathcal{SK}(\mathsf{X})}$ is an ordinary single-valued representation of $\mathcal{SK}(\mathsf{X})$. For qcb_0-spaces our two (multi-)respresentations are closely related.

[1] Note that this does not follow from the Tychonoff Theorem, because the topology of the product space may be strictly finer than the product topology.

Proposition 1 *Let* X *is be a admissibly represented* qcb_0*-space. Then:*

(1) *Any subset* K *of* X *is sequentially compact iff it is compact.*

(2) *The multirepresentations* $\delta_{\mathcal{SK}(\mathsf{X})}$ *and* $\delta_{\mathcal{K}(\mathsf{X})}$ *are topologically equivalent.*

(3) *If inequality on* X *is semi-decidable, then* $\delta_{\mathcal{SK}(\mathsf{X})}$ *and* $\delta_{\mathcal{K}(\mathsf{X})}$ *are computably equivalent representations.*

(4) *In general,* $\delta_{\mathcal{SK}(\mathsf{X})}$ *and* $\delta_{\mathcal{K}(\mathsf{X})}$ *are not computably equivalent.*

(5) *Any* $\delta_{\mathcal{SK}(\mathsf{X})}$*-computable sequence* $(K_i)_{i\in\mathbb{N}}$ *in* X *uniformly co-c.e. compact.*

It is an open question whether $\delta_{\mathcal{SK}(\mathsf{X})}$ and $\delta_{\mathcal{K}(\mathsf{X})}$ are computably equivalent, if X is computable Kolmogorov space [1], i.e. carries an effectively admissible [5, 6] representation.

A computable version of the Tychonoff Theorem

It is well-known in topology that the countable product of a sequence of sequentially compact spaces is sequentially compact again (cf. [3]). We present an effectivisation of this fact.

Theorem 2 *Let* X_i *be represented spaces for* $i \in \mathbb{N}$*. Then the function that maps a sequence* $(K_i)_{i\in\mathbb{N}}$ *of sequentially compact subsets* $K_i \in \mathcal{SK}(\mathsf{X}_i)$ *to the sequentially compact subset* $\prod_{i\in\mathbb{N}} K_i$ *in the product space* $\bigotimes_{i\in\mathbb{N}} \mathsf{X}_i$ *is computable.*

By Proposition 1, sequential compactness and compactness agree for subsets of qcb-spaces. So Theorem 2 yields an effectivisation of the sequential Tychonoff Theorem.

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Speedability as a self-translation property

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Abstract. The notion of speedability of left-c.e. reals was introduced by Merkle and Titov [4] in 2020 in terms of left approximations: a left-c.e. real α is called *speedable* if there exists a constant $\rho < 1$ and left approximation $a_0, a_1, \dots$ of α such that

$$\liminf_{n\to\infty} \frac{\alpha - a_{n+1}}{\alpha - a_n} \leq \rho. \tag{1}$$

By [4, Theorem 3], this property is robust to the choice of $\rho \in (0, 1)$. In 2024, Hölzl, Janicki, Merkle, and Stephan [2] also investigated the specific case of speedable reals fulfilling (1) with $\rho = 0$; they called such reals *superspeedable* and showed that superspeedability is strictly stronger than speedability.
Nowadays, the notion of speedability is briefly researched in the context of relative randomness (see e.g. [5], [1], [3]).

In the scope of this talk, we propose some functional characterizations of the speedability and superspeedability properties: a real α is speedable iff there exists a constant $\rho < 1$ and monotone function f, which is computable on rationals or on reals left from α and fulfills

$$\liminf_{x\nearrow\alpha} \frac{\alpha - f(x)}{\alpha - x} < \rho, \tag{2}$$

and superspeedable iff the constant ρ in (2) can be chosen equal to zero. We extend the functional characterizations to all reals and show that these extensions preserve some properties of speedability and superspeedability on left-c.e. reals; in particular, they are also robust to the choice of $\rho \in (0, 1)$. We also explain the meaning of speedability from the viewpoint of computable analysis as nonexistence of a left derivation of a computable monotone function with an appropriate domain in its fixed point.

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The Arithmetic Hierarchy of Real Functions

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Abstract

Inspired by Zheng and Weihrauch's arithmetic(al) hierarchy of real numbers, we investigate a hierarchy of computability levels for functions from $\mathbb{R} \to \mathbb{R}$. In particular, we seek to capture the classes of estimable, lower semicomputable, and upper semicomputable functions, understand how they are connected, and whether they can be generalized to important larger classes. We establish the lattice structure of our hierarchy along with its properties under various analytical and algebraic operations that are useful for calculations, particularly function composition. We also draw a connection to the arithmetic hierarchy of sets of natural numbers, extending classical results such as Shoenfield's limit lemma by allowing oracles with infinite input. Our main contribution is to make real hypercomputation results of Ziegler and Brattka accessible to computer scientists who are not computable analysts, which has direct applications to areas such as algorithmic information theory, descriptive set theory, and scientific computing.

Keywords

Arithmetic hierarchy; Computable analysis; Shoenfield's limit lemma; Real hypercomputation.

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