Robustness in Dynamical Systems: when allowing noise makes the undecidable decidable

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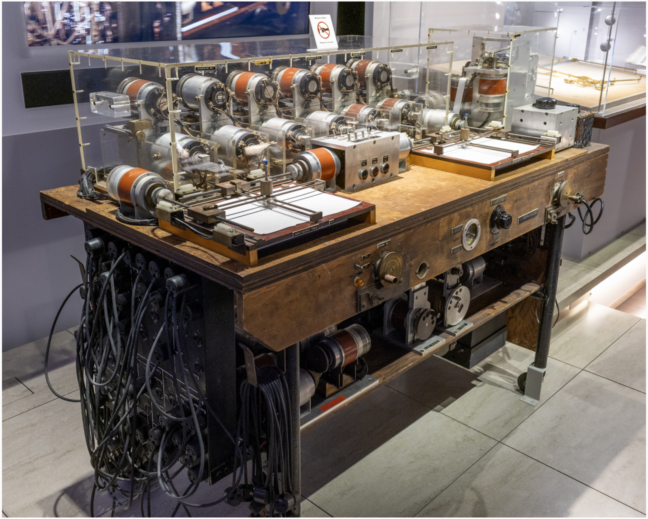
Introduction

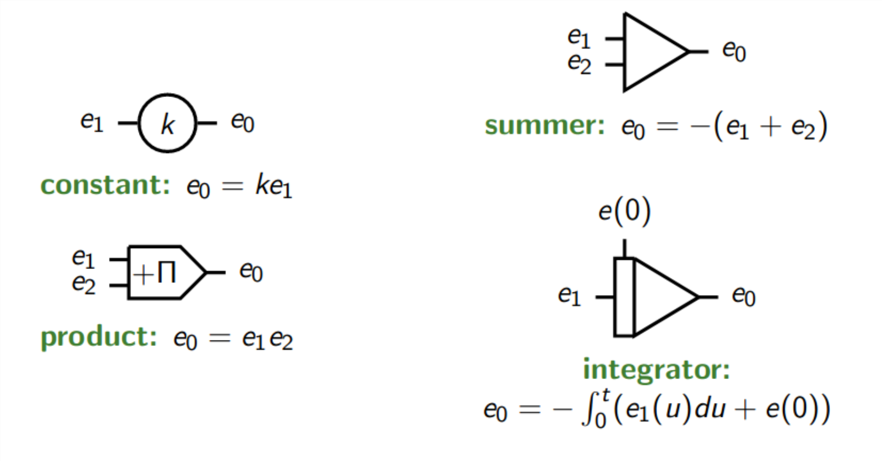
Let's start with a wee example (from analogue models)

The Differential Analyser:



Let's start with a wee example (GPAC)

The "General Purpose Analog Computer" (GPAC):



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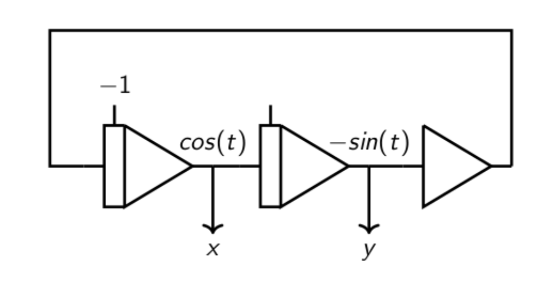

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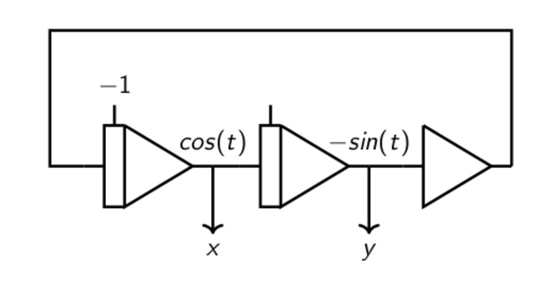

$$\begin{cases} x'(t) &= y(t) \\ y'(t) &= -x(t) \\ x(0) &= -1 \\ y(0) &= 0 \end{cases} \Rightarrow \begin{cases} x(t) &= \cos(t) \\ y(t) &= -\sin(t) \end{cases}$$

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$$\begin{cases} y' = z & y(0) = 0 \\ z' = -2uz^2 & z(0) = 1 \\ u' = 1 & u(0) = 0 \end{cases}$$

- Let the system run for 1 time unit;

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  - Any (analogue) device is subjected to pertubations, or **noise**
    $\Rightarrow$ integretors accumulate errors, the clock is not perfect, ...

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$I_\varepsilon(x) = [x - \varepsilon, x + \varepsilon]$, as $\epsilon \to 0$

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$\Rightarrow$ we can solve (almost) anything on a compact

$\Rightarrow$ Now we can potentially solve undecidable computation problems (e.g. soving ODEs)

(Real) Introduction

But how do we formalise this notion of noise?

What are the implications in computability/complexity?

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But how do we formalise this notion of noise?

What are the implications in computability/complexity?

We are first interested here in the notion of **reachability** in different contexts

- Motivation/Starting point in theoretical computer science
- In Turing Machines
- In Dynamical systems (PSPACE-completeness)

Introduction: our Starting Point

- Motivated by the fields of verification and topology.
- Informal conjecture: undecidability in verification does not happen for *robust systems*.

Turing Machines and Space Perturbations

Computability: Properties

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- It becomes decidable for a sub-class of TMs: Robust TMs.

Computability: n-perturbation of a TM

Let $\mathcal{M}_n$, the n-space-perturbed version of TM $\mathcal{M}$:

$\Rightarrow$ the n-perturbed version of the machine $\mathcal{M}$ is unable to remain correct at a distance more than n from the head.

$$\cdots \;\|\; \underbrace{l_{-n+1} \mid \cdots \mid l_{-1} \mid l_0}_{n} \;\uparrow\; \underbrace{r_0 \mid r_1 \mid \cdots \mid r_{n-1}}_{n} \;\|\; \cdots$$

Computability: Some Properties (Asarin & Bouajjani '01)

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    - $L(\mathcal{M}) \subseteq L_\omega(\mathcal{M}) \subseteq \cdots \subseteq L_2(\mathcal{M}) \subseteq L_1(\mathcal{M})$.

Computability: Perturbed reachability co-ce

Theorem (Perturbed reachability is co-c.e., from Asarin-Bouajjani 01)

$$L_\omega(\mathcal{M}) \in \Pi^0_1.$$

Say L is robust if $L = L_\omega$:

$$L \text{ robust} \Rightarrow L \text{ decidable.}$$

Definition
For $f : \mathbb{N} \to \mathbb{N}$, we write $L_{\{f\}}(\mathcal{M})$ the set of words accepted by $\mathcal{M}$ with a "space-perturbation" f:

$$L_{\{f\}}(\mathcal{M}) = \{w \mid w \in L_{f(\ell(w))}(\mathcal{M})\}.$$

Complexity: A Characterisation of PSPACE

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$\rightarrow$ the size of the window depends on the size of the input word

Complexity: Polynomial precision robust $\Leftrightarrow$ PSPACE,

Theorem (CSL24, JoC26, B., Bournez)

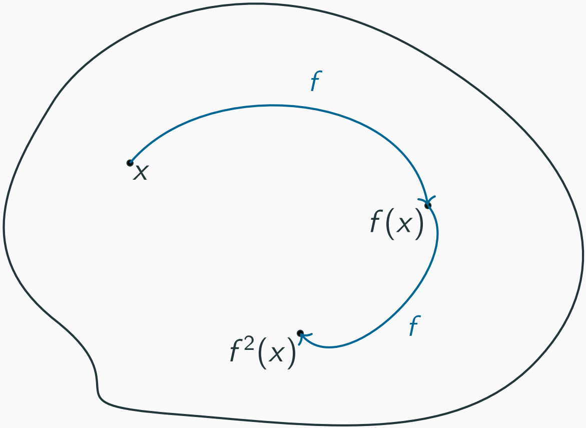
$L \in \mathrm{PSPACE}$ iff for some $\mathcal{M}$ and some polynomial p,

$$L = L(\mathcal{M}) = L_{\{p\}}(\mathcal{M}).$$

- Definition of the notion of perturbation of a TM
- Some properties
- Characterisation of PSPACE with space-perturbed TMs

Dynamical Systems and Reachability Relations

Dynamical Systems



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- It becomes decidable for a sub-class of systems: Robust Dynamical Systems.

Reachability Relation for Dynamical Systems

$\rightarrow$ With each rational discrete-time dynamical system $\mathcal{H}$ is associated its reachability relation $R^{\mathcal{H}}(\cdot, \cdot)$ on $\mathbb{Q}^d \times \mathbb{Q}^d$.

$\rightarrow$ two rational points $\mathbf{x}$ and $\mathbf{y}$, $R^{\mathcal{H}}(\mathbf{x}, \mathbf{y})$ holds iff there exists a trajectory in $\mathcal{H}$ from $\mathbf{x}$ to $\mathbf{y}$.

$\rightarrow$ $R^{\mathcal{H}}(\mathbf{x})$ if the set of all the points reachable from $\mathbf{x}$.

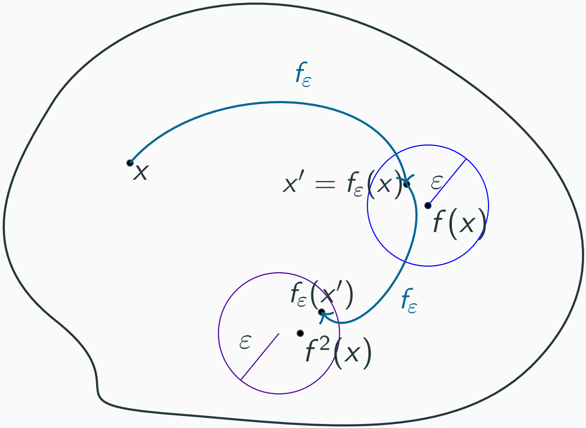
$\rightarrow$ Associated problem: $\mathfrak{R}^{\mathcal{H}}$.

Computability

- Undecidabilty is related to non-robustness of the systems (Asarin, Bouajjani)

  $\rightarrow$ with proper notion of robusteness: no sensitivity to an arbitrarly small perturbation.

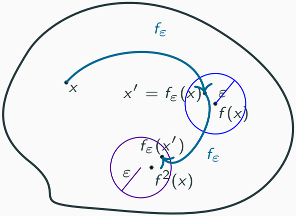
Robustness in Dynamical Systems



Reachability Relation With Small Perturbations

- Consider now a discrete-time dynamical system $\mathcal{H}$ with function $\mathbf{f}$ Lipschitz on a compact domain:
  For $\varepsilon > 0$ we consider the ε-perturbed system $\mathcal{H}_\varepsilon$.

- Its trajectories are defined as sequences $\mathbf{x}_t$ satisfying $d(\mathbf{x}_{t+1}, \mathbf{f}(\mathbf{x}_t)) < \varepsilon$ for all t.



Reachability Relation With Small Perturbations

- Reachability relation in $\mathcal{H}_\varepsilon$: $R_\varepsilon^{\mathcal{H}}(\cdot,\cdot)$.
  Associated problem: $\mathfrak{R}_\epsilon^{\mathcal{H}}$
- $R_\omega^{\mathcal{H}} = \bigcap_\epsilon R_\varepsilon^{\mathcal{H}}(\cdot,\cdot)$ is co-c.e..
- Say $\mathcal{H}$ is robust when $R_\omega^{\mathcal{H}} = R^{\mathcal{H}}$:

$$\mathcal{H} \text{ robust} \Rightarrow \mathfrak{R}^{\mathcal{H}} \text{ decidable}$$

- Reachability in robust dynamical systems is *decidable*

Theorem (CSL24, JoC26, B., Bournez)

Take a locally Lipschitz system, with $f : X \to X$ polynomial-time computable, with X a closed rational box. Then, for $p : \mathbb{N} \to \mathbb{N}$ a polynomial, $\mathfrak{R}_p^{\mathcal{H}} \subseteq \mathbb{Q}^d \times \mathbb{Q}^d \times \mathbb{N} \in \mathrm{PSPACE}$.

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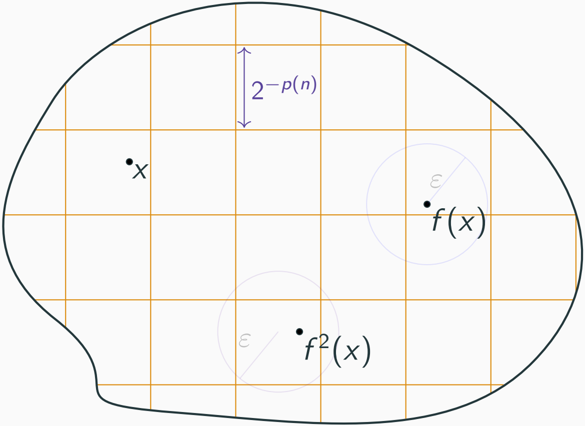
$\to$ $X = \prod_{i=1}^{n} [a_i, b_i]^d \subseteq \mathbb{R}^d$, $a_i, b_i \in \mathbb{Q}$

$\to$ works also for dynamical systems over the reals

$\to$ proof using *thickened* graphs

Proof idea (with a grid)



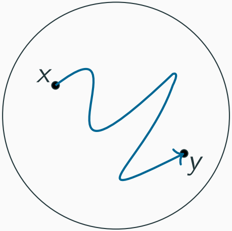
- A vertex $\equiv$ a square
- An edge $\equiv$ one computation step

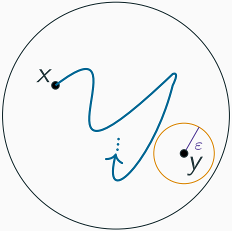
Link with δ-reachability

Proposition (Robust $\Leftrightarrow$ reach true or ϵ-far from being true)

We have $R^{\mathcal{H}}_{\omega} = R^{\mathcal{H}}$ iff for all $\mathbf{x}, \mathbf{y} \in \mathbb{Q}^d$, either $R^{\mathcal{H}}(\mathbf{x}, \mathbf{y})$ is true or $R^{\mathcal{H}}(\mathbf{x}, \mathbf{y})$ is false and there exists $\epsilon > 0$ such that it is ϵ-far from being true.

$\Rightarrow$ relation with δ-reachability (Gao, Kong, Chen, Clarke '06).





Complexity of reachability relations

Polynomially robust to precision $\Leftarrow$ PSPACE-completeness

Theorem (CSL24, JoC26, B., Bournez)

Any PSPACE *language is reducible to PAM's reachability relation:*
$R^{\mathcal{H}} = R^{\mathcal{P}}{}_{\{p\}}$, *for some polynomial p.*

Drawability and extension to the reals

Drawability: geometric properties

$\Rightarrow$ we can also see it as a relation over the reals and use the framework of computable analysis

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- What is a computable real number x?

  There exists a Cauchy sequence $\phi : \mathbb{N} \to \mathbb{D}$ such that for all $n \in \mathbb{N}$, $|\phi(n) - x| \leq 2^{-n}$.

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- What is a computable function $f : \mathbb{R} \to \mathbb{R}$?
  There exists an oracle Turing machine M such that, on a Cauchy sequence ϕ converging to x, M queries the oracle to have m such that $|\phi(m) - x| \leq 2^{-m}$ and computes $d \in \mathbb{D}$ such that $|d - f(x)| \leq 2^{-n}$.

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Definition
A closed subset of $\mathbb{R}^d$ is said *c.e. closed* if we can effectively enumerate open balls of rational radii intersecting it.

Drawability: geometric properties

From the statements of CA, the following holds:

Theorem

Consider a computable discrete time system $\mathcal{H}$ whose domain is a computable compact. For all $\mathbf{x}$, *$cls(R^{\mathcal{H}}(x)) \subseteq \mathbb{R}^d$ is a c.e. closed subset.*

And With Reachability Properties

Theorem (Perturbed reachability is co-c.e.)

Consider a dynamical system, with $\mathbf{f}$ *locally Lipschitz, computable, whose domain is a computable compact, then, for all* $\mathbf{x}$*,* $cls(R_\omega^{\mathcal{H}}(\mathbf{x})) \subseteq \mathbb{R}^d$ *is a co-c.e. closed subset.*

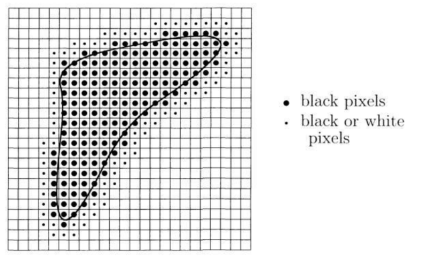
Say $\mathcal{H}$ is robust when $R_\omega^{\mathcal{H}} = R^{\mathcal{H}}$:

$$\mathcal{H} \text{ robust} \Rightarrow \mathfrak{R}^{\mathcal{H}} \text{ decidable}$$

$\rightarrow$ proof with *thickened* graphs

Plot twist

From CA, Computable $\Leftrightarrow$ can be plotted.



Theorem (CSL24, JoC26, B., Bournez)
$R^{\mathcal{H}}$ closed and can be plotted in a name of $\mathbf{f}$ $\Leftrightarrow$ the system is robust, i.e. $R_{\omega}^{\mathcal{H}} = R^{\mathcal{H}}$.

Partial Conclusion

- Perturbed reachability relation for rational dynamical systems
- Characterisation of PSPACE in that framework (with additionnal properties, including L-Lipchitz)
- We have a nice extension to geometric properties : what can be drawn is robust

- Outer ε-perturbation: $R^{\varepsilon}(\mathbf{x}) := \overline{B}\big(R(\overline{B}(\mathbf{x}, \varepsilon)), \varepsilon\big)$.
- The discretisation error is controlled by L:

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- **Limitations:** Defined *only* for deterministic, Lipschitz and continuous systems.

Allowing Nondeterminism: (Effective) Topological Generalisations

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  $\Rightarrow$ explained by the role of topology in global properties.
- Also, we considered only reachability in deterministic system, with a Lipschitz dynamic

Generalisation: Classical Properties seen as Fixed Points

Application: Lfp/gfp in our theory

Let $\Gamma : \mathcal{P}(X) \to \mathcal{P}(X)$ be a monotone operator.

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- A set $Y \subseteq X$ is said to be *prefixed* for Γ if $\Gamma(Y) \subseteq Y$.
  - $\mathrm{Pre}(\Gamma) = \{ Y \subseteq X : \Gamma(Y) \subseteq Y \}$ the family of all prefixed points of Γ.
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- A set $Y \subseteq X$ is said to be *postfixed* for Γ if $Y \subseteq \Gamma(Y)$.
  - $\mathrm{Post}(\Gamma) = \{ Y \subseteq X : Y \subseteq \Gamma(Y) \}$ the family of all postfixed points of Γ.
  - Then the gfp of Γ satisfies $\mathrm{gfp}(\Gamma) = \bigcup \mathrm{Post}(\Gamma)$

Various properties as fixed points

Many concepts from control and verification can be cast as fixed points. Let $R : X \rightrightarrows X$ be a (possibly non-deterministic) dynamical system, $K \subseteq X$.

Concept	Operator $\Gamma(S)$
Reachability from x	lfp of $S \mapsto \{x\} \cup R(S)$

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Viability kernel of K	gfp of $S \mapsto K \cap \{x : R(x) \cap S \neq \emptyset\}$
Bisimulation	gfp on $\mathcal{P}(X \times X)$

Back to local vs. global

By definition, every monotone operator admits a canonical witness, namely its fixed points $\mathrm{lfp}(F)$ and $\mathrm{gfp}(F)$ themselves.

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- *Local* (reachability, inevitability) $\rightsquigarrow$ lfp.
- *Global* (invariance, viability, bisimulation) $\rightsquigarrow$ gfp.

Constructive witness of fixed points

The key issue is not the existence of such witnesses, which is guaranteed abstractly, but their effective accessibility:

Let $F : \mathcal{P}(X) \to \mathcal{P}(X)$ be monotone.

- If $Z \subseteq X$ is *post-fixed*, i.e. $Z \subseteq F(Z)$, then $Z \subseteq \mathrm{gfp}(F)$.
  $\to x \in Z$ is a constructive witness that $x \in \mathrm{gfp}(F)$.
- If $Z \subseteq X$ is *pre-fixed*, i.e. $F(Z) \subseteq Z$, then $\mathrm{lfp}(F) \subseteq Z$.
  $\to x \notin Z$ is a certificate that $x \notin \mathrm{lfp}(F)$.

What we did, and where we go

Until now:

- reachability (one lfp),
- deterministic systems,
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Short answer: yes, with the right topological setting.

Hausdorff distances and Continuity

Given a metric space (X, d) and two subsets $A, B \subseteq X$:

- *Directed Hausdorff Distance* from A to B:
  $h^{\rightarrow}(A, B) = \sup_{\mathbf{x} \in A} \; \inf_{\mathbf{y} \in B} \; d(\mathbf{x}, \mathbf{y}) = \sup_{\mathbf{x} \in A} d_x(B)$
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- *Local Continuity Modulus*: there exists a nondecreasing function $\omega_x : [0, r_0] \to [0, \infty)$ with $\omega_x(r) \to 0$ as $r \to 0$ such that, for all $\mathbf{x}'$ near $\mathbf{x}$: $h(R(\mathbf{x}'), R(\mathbf{x})) \leq \omega_x(d_X(\mathbf{x}', \mathbf{x}))$

Main Result of this Section

For all sufficiently small $0 < \delta < r_0$, for all $x \in K$,

$$R^{\omega_K(\delta)+\delta}(x) \subseteq R^{\delta}_{\mathrm{graph}}(x) \subseteq R^{\delta+\rho(\delta)}(x)$$

$\rightarrow$ generalises the Lipschitz setting of the previous section (ω in place of L).

$\rightarrow$ accommodates discontinuous f via its set-valued closure.

Theorem (Equality of ω-limits, B. & Bournez)
Let $R^{\mathcal{H}}_{\omega}(x) := \bigcap_{\varepsilon>0} R^{\mathcal{H}}_{\varepsilon}(x)$. Fix a decreasing sequence $(\delta_n)_{n\in\mathbb{N}}$ with $\delta_n \downarrow 0$, such that δ_n is small enough for above to apply and $\delta_n + \rho(\delta_n) \to 0$. Define the graph ω-limit along this sequence by $R^{\omega}_{\mathrm{graph}}(x) \ := \ \bigcap_{n\in\mathbb{N}} R^{\delta_n}_{\mathrm{graph}}(x)$.

Then, for all $x \in K$,

$$R^{\mathcal{H}}_{\omega}(x) \ = \ R^{\omega}_{\mathrm{graph}}(x)$$

Robust Fixed Points

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  $\rightarrow$ Their membership can be decided from finite information away from a thin boundary band,
  $\rightarrow$ and their approximants admit finite representations at any prescribed resolution.

Partial Conclusion

- We have a more general theory
  $\Rightarrow$ Now include also non-deterministic and/or discontinuous systems
- We have an application to fixed-point computation

Conclusion

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- We studied what makes reachability decidable for robust dynamical systems and prove that space complexity = precision;

- We defined a more general framework coming from effective hyperspace topology with various applications:
  - Previous setting (dynamical systems) (the quantitative ingredient is not L but a modulus of continuity ω)
  - Effective fixed-point computation (e.g. PSPACE-membership result)
  - Etc.

Bonus!!! Effective topology: Application to lfp and gfp

Upper and Lower Semicontinuity: relations as maps

Let R be a set-valued map $R : X \rightrightarrows Y$:

- When $R(\mathbf{x})$ is compact, R is upper semicontinuous at $\mathbf{x}$ iff $\forall \varepsilon > 0, \exists \eta > 0$ such that $\forall \mathbf{x}' \in \mathrm{Dom}(R) \cap \overline{B}_X(\mathbf{x}, \eta)$

$$R\left(\mathbf{x}'\right) \subset \overline{B}_Y(R(\mathbf{x}), \varepsilon)$$

- R is lower semicontinuous at $\mathbf{x} \in \mathrm{Dom}(R)$ iff for any $\mathbf{y} \in R(\mathbf{x})$ and for any sequence $(\mathbf{x}_n)_{n\in\mathbb{N}} \in \mathrm{Dom}(R)^{\mathbb{N}}$, $\mathbf{x}_n \to \mathbf{x}$, there exists a sequence $(\mathbf{y}_n)_{n\in\mathbb{N}} \in R\left(\mathbf{x}_n\right)^{\mathbb{N}}$ converging to $\mathbf{y}$.

Definition (Perturbed (outer) relation)

Let (X, d) be a metric space and let $R : X \rightrightarrows X$ be a set-valued map. For $\epsilon > 0$, the *outer ϵ-perturbation* of R is the set-valued map

$$R^{\varepsilon}(x) := \overline{B}\big(R(\overline{B}(x, \varepsilon)), \varepsilon\big)$$

Equivalently, $R^{\varepsilon}(x) = \bigcup_{x' \in \overline{B}(x,\varepsilon)} \overline{B}\big(R(x'), \varepsilon\big)$.

Thickened Graph

Notation: $\text{Graph}(R) = \{(\mathbf{x}, \mathbf{y}) \in X \times Y : \; \mathbf{y} \in R(\mathbf{x})\}$

Definition
For $\rho(\delta) > 0$, define the thickened graph

$$G_{\rho(\delta)} := cls(\overline{B}\big(\text{Graph}(R), \rho(\delta)\big)) \; \subseteq \; X \times Y,$$

→ Equivalently, $(u, v) \in G_{\rho(\delta)}$ iff there exists $(x', y') \in \text{Graph}(R)$ such that $d_X(u, x') \leq \rho(\delta)$ and $d_X(v, y') \leq \rho(\delta)$.

→ In particular, $\text{Graph}(R) \subseteq G_{\rho(\delta)}$.

Abstract discretisation graph

Definition (Abstract discretisation graph)

Let $G_\delta = (V_\delta, \to_\delta)$ be a directed graph, such that $V_\delta = \{B_1, \ldots, B_N\}$ and there is an edge $B_i \to_\delta B_j$ iff $(B_i \times B_j) \cap G_{\rho(\delta)} \neq \emptyset$.

Define the associated set-valued map $R^\delta_{\mathrm{graph}} : K \rightrightarrows X$ by

$$R^\delta_{\mathrm{graph}}(x) := \bigcup \Big\{ B_j \;\Big|\; \exists i \; (x \in B_i \text{ and } B_i \to_\delta B_j) \Big\}.$$

Intuitively, $R^\delta_{\mathrm{graph}}$ collects all successors compatible with the thickened graph $G_{\rho(\delta)}$ on the finite grid of boxes.