Effective decompositions of continua with embedded arcs

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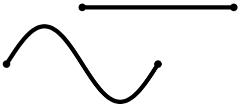
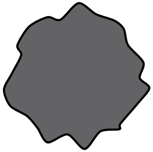
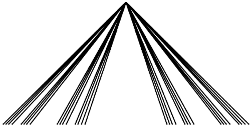
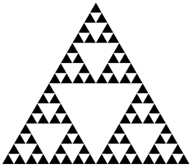
Continua

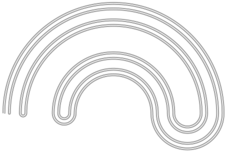
A continuum is a compact and connected metric space (with more than one point).











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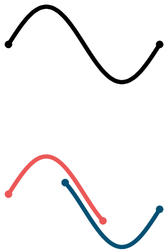

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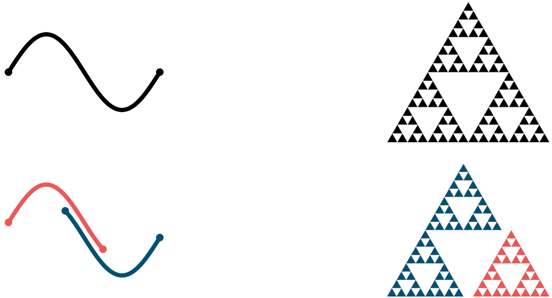

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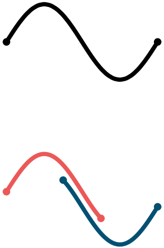

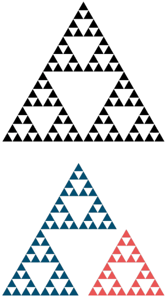

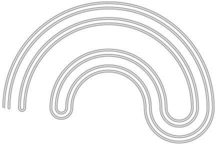

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The buckethandle: a closer look

- Connecting symmetric points in the Cantor set

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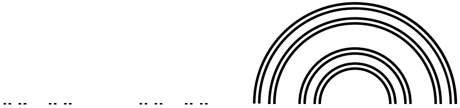

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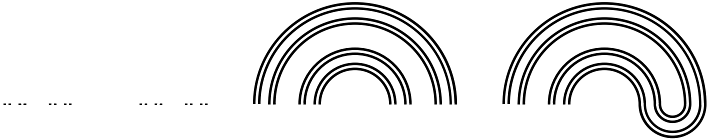

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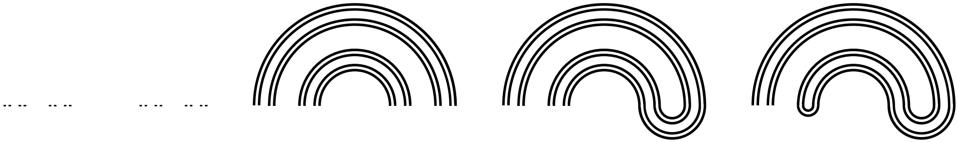

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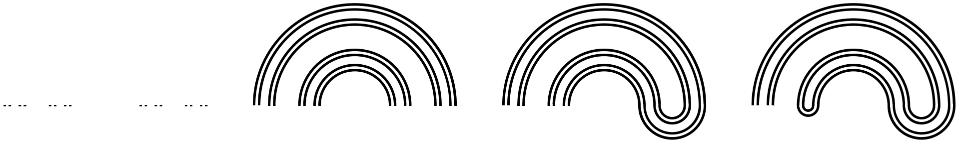
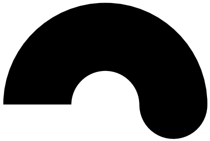


- "Digging channels" through an arc



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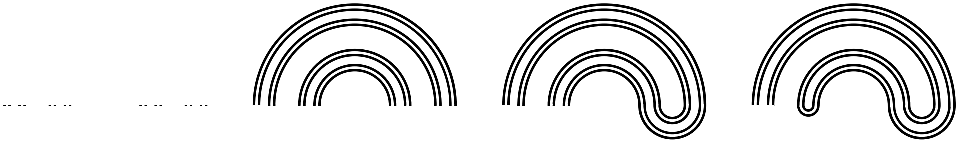

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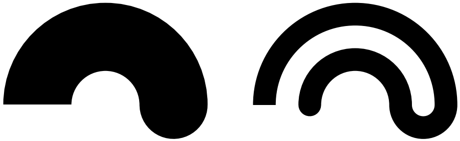

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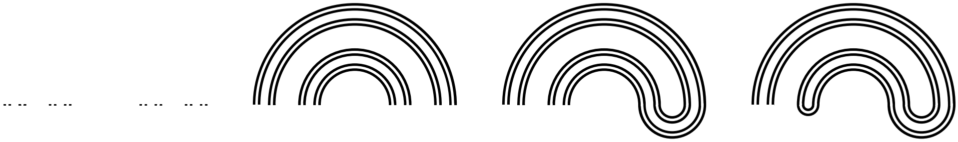

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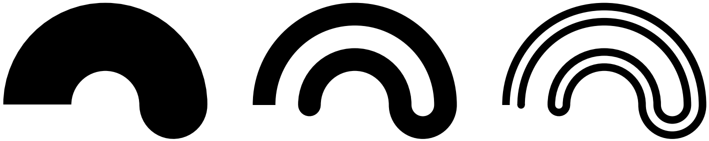

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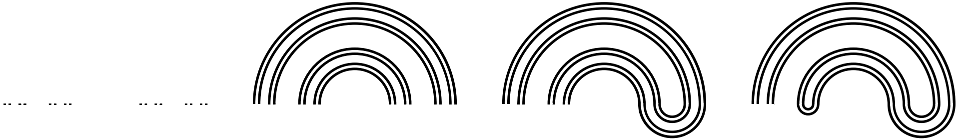

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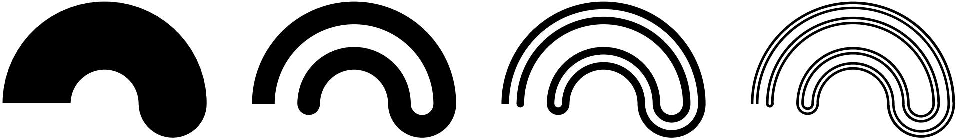

What about computability?

If a decomposable continuum is computably presented, can its decomposability be witnessed by computable subcontinua?

$\rightsquigarrow$ we call continua with this property effectively decomposable

Alternatively: is there a computably presented, decomposable continuum such that all of its decompositions encode the Halting set in some way? How much information can be encoded by (in)decomposability?

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The setting

- Computable metric space (X, d, α): metric space (X, d) + $\{\alpha_i\}_{i \in \mathbb{N}}$ dense subset such that $(i, j) \mapsto d(\alpha_i, \alpha_j)$ is computable
- Basic open balls $B(\alpha_i, q)$, $q \in \mathbb{Q}^+$

A compact set S in (X, d, α) is computable if there is a computable function $k \mapsto (i_1, i_2, \ldots, i_{n_k})$ such that $d_H(S, \{\alpha_{i_1}, \ldots, \alpha_{i_{n_k}}\}) < 2^{-k}$.

- X itself generally need not be a computable set
- If X is computable, we say that (X, d, α) is an effectively compact computable metric space

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The right kind of presentation

Back to our question: is a decomposable continuum with the structure of a computable metric space always decomposable into computable subcontinua? The answer is no.

If X is a union of its computable subcontinua, then X is itself computable. So (X, d, α) should be an effectively compact computable metric space.

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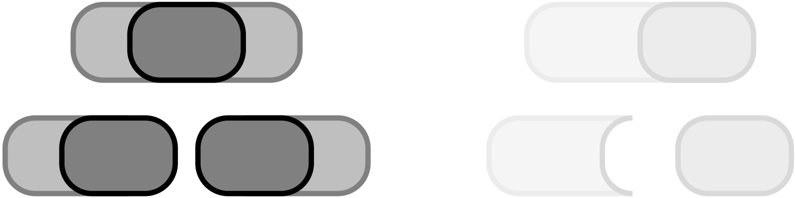

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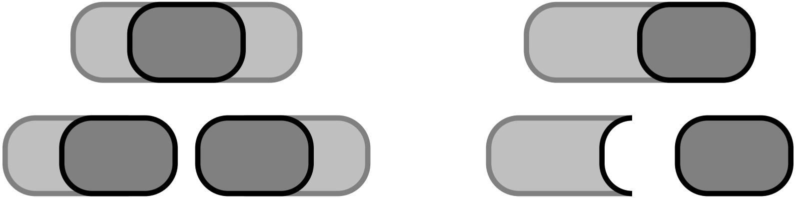

A simpler problem

Suppose an effectively compact continuum X has a computable subcontinuum C with non-empty interior.

- If $X \setminus \operatorname{Int} C$ is disconnected, the classical argument is easily effectivized
- If $X \setminus \operatorname{Int} C$ is connected, the construction fails: ideally, we want $X \setminus \operatorname{Int} C$ to be computable, but this does not have to be the case

- $X \setminus \operatorname{Int} C$ will be computable if $\operatorname{Int} C$ is c.e. open
- What also works: $\operatorname{Int} C$ contains a (suitable) c.e. open set; some finite union of basic open balls is connected; X is effectively locally connected

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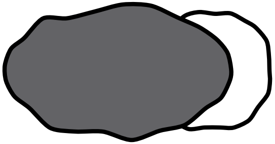
Continua containing a copy of $\mathbb{R}$

- We consider continua (X, d) which contain a copy of $[0, 1]$ as a subcontinuum with non-empty interior
- Equivalently, they contain a copy of $\mathbb{R}$ as an open subset
- Such continua are classically decomposable



- If a point has an open neighbourhood $N \cong \mathbb{R}$ in X, then it has a computable neighbourhood $N' \subseteq N$ which is an arc with computable endpoints
  (V. Čačić, Č, M. Horvat, and Z. Iljazović (2026). "Computable Approximations of Semicomputable Graphs". In: *Logical Methods in Computer Science* 22, 28 (1))

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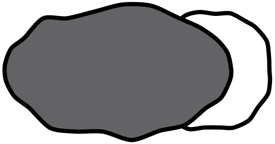

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Theorem 1 (Č, Iljazović, Jelić, Tarandek)

Let (X, d, α) be an effectively compact computable metric space. Suppose U is an open set in (X, d) homeomorphic to $\mathbb{R}$. Then (X, d, α) is effectively decomposable.

Sketch of proof.

Take a computable arc A in U; its interior is not empty.

If $X \setminus \operatorname{Int} A$ is disconnected, i.e. $X \setminus \operatorname{Int} A = F \sqcup G$, then $A \cup F$ and $A \cup G$ are computable.

If $X \setminus \operatorname{Int} A$ is connected, take a neighbourhood N of A consisting of basic open balls; then $X \setminus N$ is computable (but possibly not connected).

Take two more computable arcs A_1, A_2 such that $A_1 \cup (X \setminus N) \cup A_2$ becomes connected (and computable as a union of computable sets).

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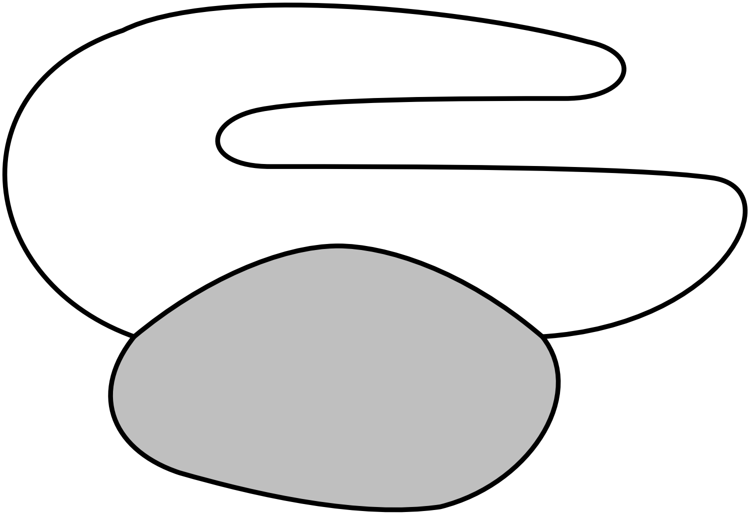

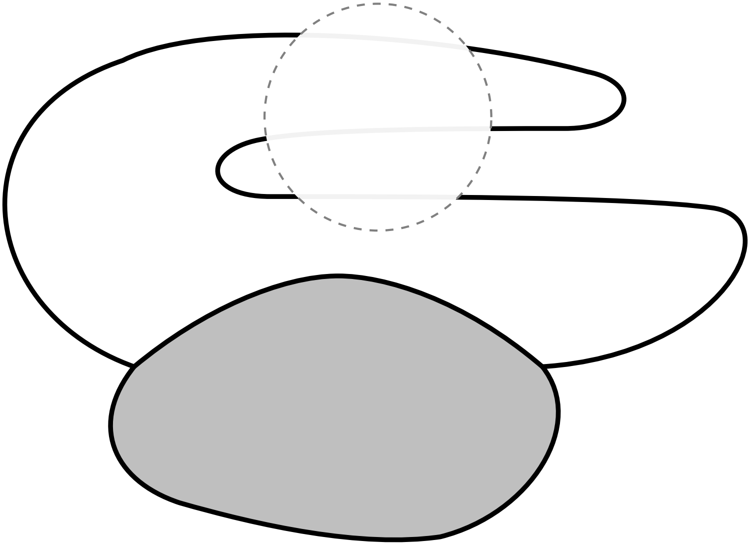

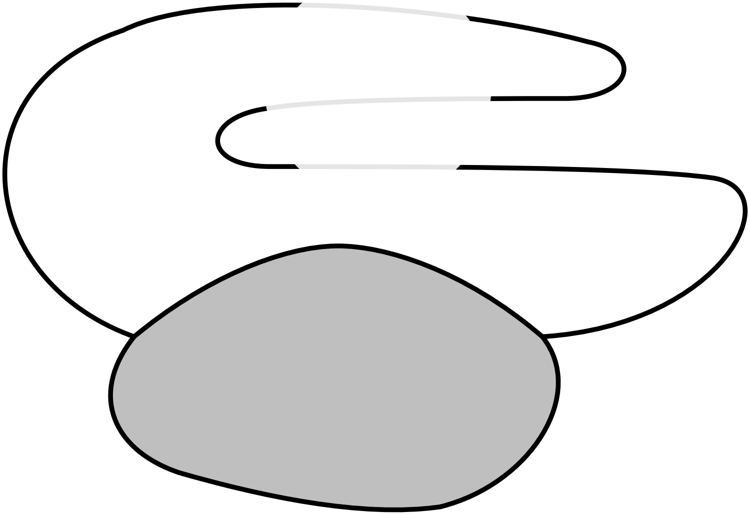

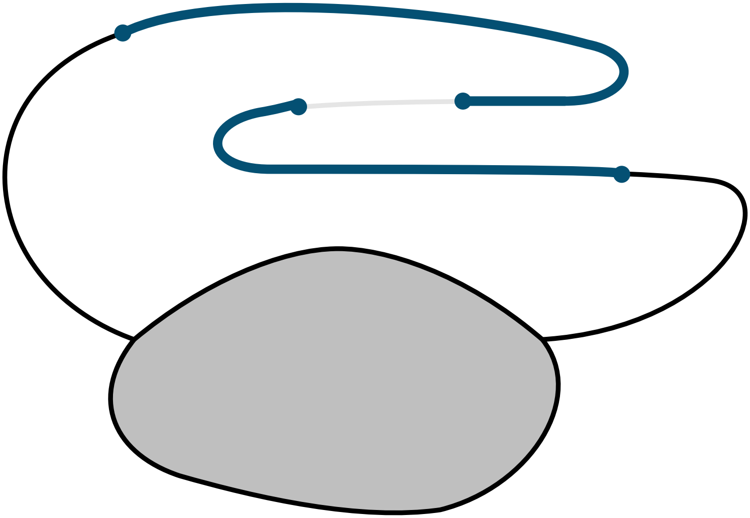

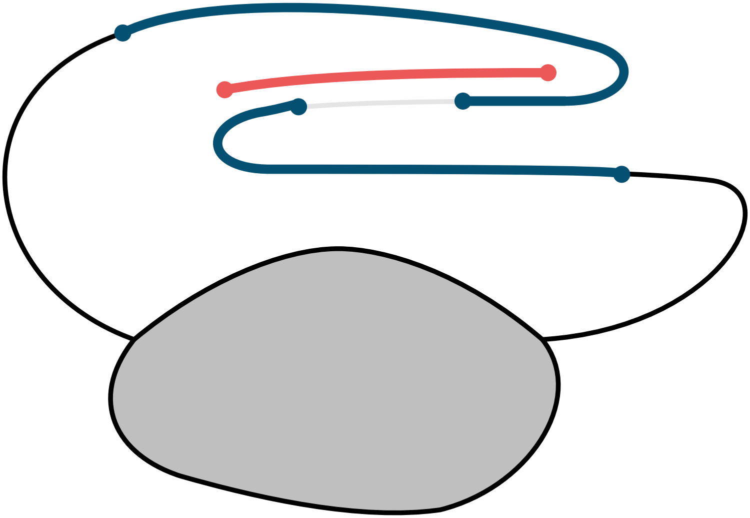

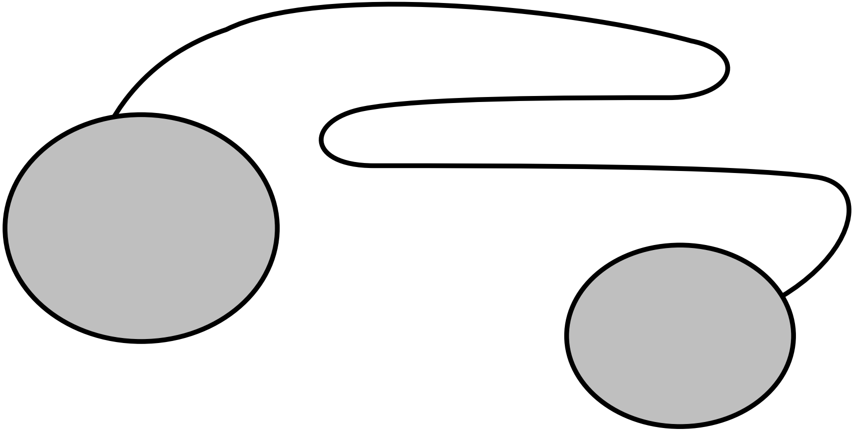

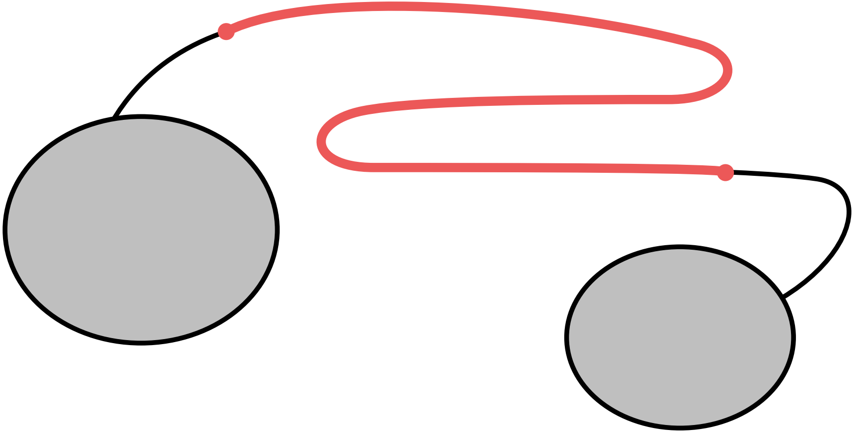

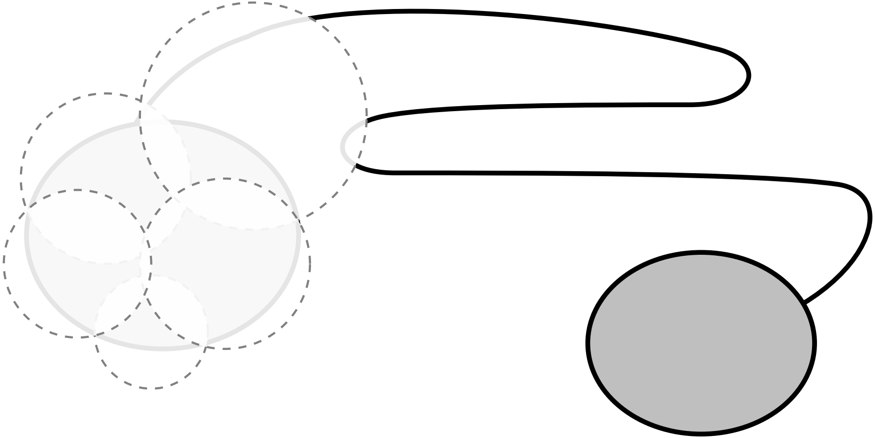

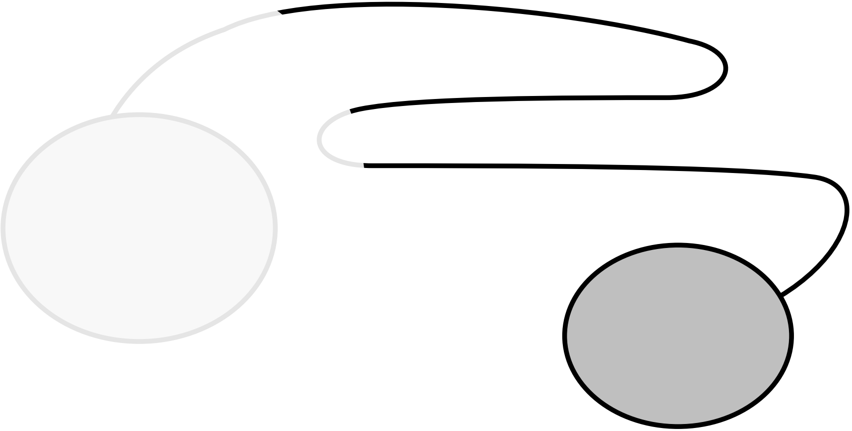

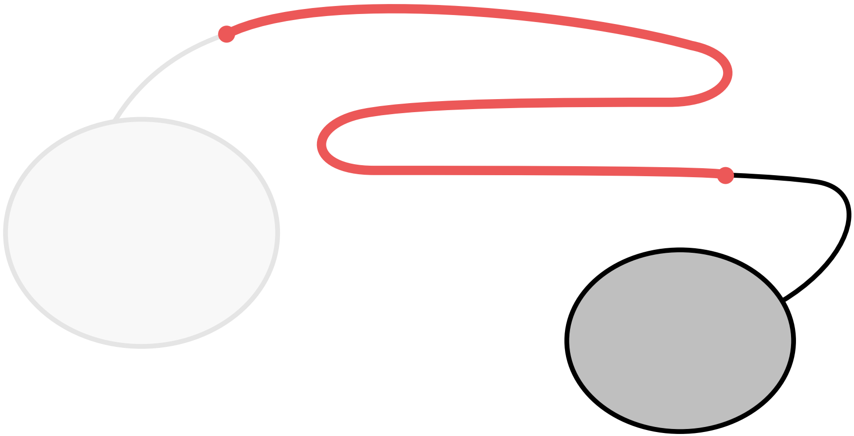

Decomposable chainable continua

- Chainable continuum: for each $\varepsilon > 0$ can be covered by an open chain $U_1, \ldots, U_n$ such that $\max \operatorname{diam} U_i < \varepsilon$.
- (Computable) chainable continua can be indecomposable (buckethandle, pseudo-arc)

Theorem 2 (Č., Iljazović, Jelić, Tarandek)

Let (X, d, α) be an effectively compact computable metric space such that (X, d) is a decomposable chainable continuum. Then (X, d, α) is effectively decomposable.

Proof. Technical. Uses techniques similar to those developed in

Z. Iljazović (2009). "Chainable and Circularly Chainable Co-c.e. Sets in Computable Metric Spaces". In: *Journal of Universal Computer Science* 15.6, pp. 1206–1235.

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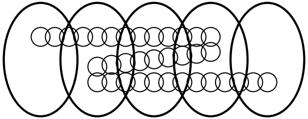

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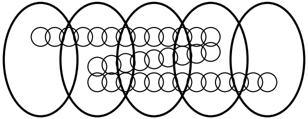

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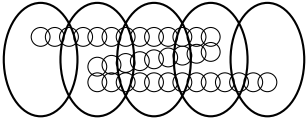

Theorem 2 (Č., Iljazović, Jelić, Tarandek)

Let (X, d, α) be an effectively compact computable metric space such that (X, d) is a decomposable chainable continuum. Then (X, d, α) is effectively decomposable.

Proof. Technical. Uses techniques similar to those developed in

Z. Iljazović (2009). "Chainable and Circularly Chainable Co-c.e. Sets in Computable Metric Spaces". In: *Journal of Universal Computer Science* 15.6, pp. 1206–1235.

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Thank you for your attention!

Questions? Comments?

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