Effective subcategories of quasi-Polish spaces

Matthew de Brecht[1]

Kyoto University

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Introduction

- In previous work, we constructed the category of quasi-Polish spaces (QPol) as a represented space, and showed that limits, coproducts, and standard powerspace monads are computable in the sense of Type Two Theory of Effectivity (TTE).
- In this talk, we show that some important sub-categories of QPol (e.g., overt discrete quasi-Polish spaces (ODS) and compact Hausdorff quasi-Polish spaces (CHS)) can be constructed as effective quasi-Polish spaces (i.e., effective internal categories of QPol).
- To show that the constructions are natural, we show that Stone duality (for both objects and morphisms) is computable, in the sense that the dual contravariant functors and the natural isomorphisms demonstrating their equivalence are computable.
    - This generalizes recent work by N. Bazhenov et al. (2023) on computable Stone spaces, and old work by S. Odintsov & V. Selivanov (1989) on enumerated Boolean algebras.

Motivation

1. Quasi-Polish categories
   - Recent work by Ruiyuan Chen used (continuous actions of) quasi-Polish groupoids of countable models to represent syntactic ω_1-pretoposes of ω_1-coherent (Π_2) theories.
     - This is very similar to classifying toposes for geometric theories.
   - The quasi-Polish groupoids of models he used can be viewed as subcategories of ODS.
   - Chen's work is in the intersection of Grothendieck topos theory, invariant descriptive set theory, and countable model theory. We expect our work will be useful in effectivizing Chen's work.
     - R. Heckmann showed that quasi-Polish spaces are equivalent to countably presented locales, and there are deep connections between locale theory and Grothendieck toposes (such as the Joyal-Tierney theorem).

Motivation

2. Computability theory
   - ODS and its subcategories can be used for developing countable (computable) model theory (including c.e. presented structures) in a uniform way.
   - CHS and its subcategories can be used for a uniform treatment of computability on some important non-countable structures, such as compact Polish groups (A. Melnikov).
     - Classically, CHS is equivalent to the category of compact Polish spaces.
   - Other connections between category theory and computability:
     - R. Miller et al. studied computable functors and computable equivalence of categories of structures on $\mathbb{N}$.
     - Recent work by T. Kihara connecting degree theory and Lawvere-Tierney topologies on the effective topos.
     - J. Franklin et al. recently used represented spaces of represented spaces, constructed as fiber bundles over a base powerspace, to study computable categoricity of compact Polish spaces.

Motivation

3. Foundations

- By encoding quasi-Polish spaces as spaces of ideals, many techniques in C. Mummert's PhD thesis on the reverse mathematics of general topology can be used in our setting.
- In particular, most (if not all) of the results in this talk can be carried out within ACA_0.
  - We leave a more careful analysis for future research.

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Quasi-Polish spaces

A topological space is quasi-Polish iff it satisfies any of the following equivalent properties:

- It is a countably based space with a topology generated by a (Smyth-) complete quasi-metric
- It is homeomorphic to a $\mathbf{\Pi}^0_2$-subspace of $\mathcal{P}(\mathbb{N})$, the powerset of the natural numbers with the Scott-topology.
  - A subset S is $\mathbf{\Pi}^0_2$ iff there are sequences $(U_i)_{i\in\mathbb{N}}$ and $(V_i)_{i\in\mathbb{N}}$ of opens such that $x \in S \iff (\forall i \in \mathbb{N})\,[x \in U_i \Rightarrow x \in V_i]$.
- It is a T_0-space and the image of a Polish space under an open continuous function
- and more...

Polish $\iff$ countably based & completely metrizable.

Quasi-Polish $\iff$ countably based & completely quasi-metrizable.

(Fact: Polish = quasi-Polish + metrizable)

Spaces of Ideals

Definition

Let $\prec$ be a transitive relation on $\mathbb{N}$. A subset $I \subseteq \mathbb{N}$ is an ideal (with respect to $\prec$) if and only if:

1. $I \neq \emptyset$, *(I is non-empty)*
2. $(\forall a \in I)(\forall b \in \mathbb{N})\,(b \prec a \Rightarrow b \in I)$, *($I$ is a lower set)*
3. $(\forall a, b \in I)(\exists c \in I)\,(a \prec c \,\&\, b \prec c)$. *($I$ is directed)*

The collection $\mathbf{I}(\prec)$ of all ideals has the topology generated by basic open sets of the form $[n]_\prec = \{I \in \mathbf{I}(\prec) \mid n \in I\}$ for $n \in \mathbb{N}$.

Theorem (M. d, A. Pauly, & M. Schröder, 2019)

A space is quasi-Polish if and only if it is homeomorphic to a space of the form $\mathbf{I}(\prec)$ for some transitive relation $\prec$ on $\mathbb{N}$.

Within TTE, the standard admissible representation of $\mathbf{I}(\prec)$ is given by defining a name for $I \in \mathbf{I}(\prec)$ to be any enumeration of its elements.

Examples

Example

If $=$ is the equality relation on $\mathbb{N}$, then $\mathbf{I}(=)$ is homeomorphic to $\mathbb{N}$ with the discrete topology.

We also consider relations on other countable sets (encoded by $\mathbb{N}$)

Example

If $\prec$ is the strict prefix relation on the set $2^{<\mathbb{N}}$ of finite binary sequences, then $\mathbf{I}(\prec)$ is homeomorphic to the Cantor space $2^{\mathbb{N}}$.

Example

If $\subseteq$ is the usual subset relation on the set $\mathcal{P}_{\text{fin}}(\mathbb{N})$ of finite subsets of $\mathbb{N}$, then $\mathbf{I}(\subseteq)$ is homeomorphic to $\mathcal{P}(\mathbb{N})$, the powerset of the natural numbers with the Scott-topology.

Examples: Completion of separable metric spaces

- Let (X, d) be a separable metric space. Fix a countable dense subset $D \subseteq X$, and define a transitive relation $\prec$ on $P = D \times \mathbb{N}$ as

$$\langle x, n\rangle \prec \langle y, m\rangle \iff d(x, y) < 2^{-n} - 2^{-m}.$$

  - This definition guarantees that the open ball with center x and radius 2^{-n} contains the closed ball with center y and radius 2^{-m}.
- $\mathbf{I}(\prec)$ is homeomorphic to the completion of (X, d).
  - This is related to the *formal ball* models in domain theory.
  - (Note that the metrizable quasi-Polish spaces are precisely the Polish spaces)

Continuous functions

Definition

Let $\prec_1$ and $\prec_2$ be transitive relations on $\mathbb{N}$.

- A code for a partial function is any subset $R \subseteq \mathbb{N} \times \mathbb{N}$.
- Each code R represents the partial function $\ulcorner R \urcorner :\subseteq \mathbf{I}(\prec_1) \to \mathbf{I}(\prec_2)$ defined as

$$\begin{aligned} \ulcorner R \urcorner(I) &= \{n \in \mathbb{N} \mid (\exists m \in I)\, \langle m, n\rangle \in R\}, \\ dom(\ulcorner R \urcorner) &= \{I \in \mathbf{I}(\prec_1) \mid \ulcorner R \urcorner(I) \in \mathbf{I}(\prec_2)\}. \end{aligned}$$

Theorem

A total function $f\colon \mathbf{I}(\prec_1) \to \mathbf{I}(\prec_2)$ is continuous if and only if there is a code $R \subseteq \mathbb{N} \times \mathbb{N}$ such that $f = \ulcorner R \urcorner$.

Spaces of Ideals

Definition

A space of the form $\mathbf{I}(\prec)$ for a computably enumerable (c.e.) transitive relation $\prec$ is called an effective quasi-Polish space.

- They are also called precomputable quasi-Polish spaces.
- Effective quasi-Polish spaces are closed under Π^0_2-subspaces.

Theorem

Let $\prec_1$ and $\prec_2$ be c.e. transitive relations. Then a total function $f \colon \mathbf{I}(\prec_1) \to \mathbf{I}(\prec_2)$ is computable if and only if there is a c.e. code $R \subseteq \mathbb{N} \times \mathbb{N}$ such that $f = \ulcorner R \urcorner$.

- Computability is defined in the TTE sense.
- Intuitively, $f \colon \mathbf{I}(\prec_1) \to \mathbf{I}(\prec_2)$ is computable if and only if there is an algorithm that enumerates $f(I) \in \mathbf{I}(\prec_2)$ given an enumeration of $I \in \mathbf{I}(\prec_1)$.

Lower ($\mathbf{A}(\cdot)$) and Upper ($\mathbf{K}(\cdot)$) Powerspaces

Definition

Let X be a topological space.

- $\mathbf{A}(X)$ is the set of closed (overt) subsets of X with the lower Vietoris topology, which has basic opens $\Diamond U = \{A \in \mathbf{A}(X) \mid A \cap U \neq \emptyset\}$, for open $U \subseteq X$.
- $\mathbf{K}(X)$ is the set of saturated compact subsets of X with the upper Vietoris topology, which has basic opens $\Box U = \{K \in \mathbf{K}(X) \mid K \subseteq U\}$, for open $U \subseteq X$.

Theorem

Let $\prec$ be a transitive relation on $\mathbb{N}$. Define $\prec_L$ and $\prec_U$ as:

- $A \prec_L B \iff (\forall a \in A)(\exists b \in B)\, a \prec b$ for $A, B \in \mathcal{P}_{\text{fin}}(\mathbb{N})$.
- $A \prec_U B \iff (\forall b \in B)(\exists a \in A)\, a \prec b$ for $A, B \in \mathcal{P}_{\text{fin}}(\mathbb{N})$.

Then $\mathbf{I}(\prec_L) \cong \mathbf{A}(\mathbf{I}(\prec))$ and $\mathbf{I}(\prec_U) \cong \mathbf{K}(\mathbf{I}(\prec))$. In particular, if X is an effective quasi-Polish space, then so is $\mathbf{A}(X)$ and $\mathbf{K}(X)$.

Computable topological spaces

Definition

A (countably based) computable topological space is a tuple (X, φ, S) where:

1. X is a T_0-space,
2. $\varphi\colon \mathbb{N} \to \mathbf{O}(X)$ is an enumeration of a basis for X,
3. $S \subseteq \mathbb{N}^3$ is a c.e. set satisfying $\varphi(n) \cap \varphi(m) = \bigcup\{\varphi(k) \mid \langle n, m, k\rangle \in S\}$ for each $n, m \in \mathbb{N}$.

- Computable topological spaces have been investigated by many authors.
- However, the only effective aspect of this definition is the c.e. set S.
- In particular, if (X, φ, S) is a computable topological space, then so is (Y, ψ, S) for any $Y \subseteq X$ and the corresponding restriction ψ of φ.

Completion of computable topological spaces

The following removes that ambiguity (up to computable isomorphism):

Definition

Let $S \subseteq \mathbb{N}^3$ be c.e. A computable topological space (X, φ, S) is complete if and only if for **any** computable topological space (Y, ψ, S) there is a **unique** computable embedding $e \colon Y \to X$ satisfying $\psi = (e^{-1}) \circ \varphi$.

Theorem

The complete computable topological spaces are exactly the effective quasi-Polish spaces. Precisely:

- For every c.e. subset $S \subseteq \mathbb{N}^3$, there is a complete computable topological space (X, φ, S) with X an effective quasi-Polish space.
- Conversely, every effective quasi-Polish space X is a complete computable topological space with respect to some c.e. $S \subseteq \mathbb{N}^3$.

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Effective quasi-Polish categories

Quasi-Polish categories are defined as categories internal to the category of quasi-Polish spaces (QPol) in the usual way.

Definition

A quasi-Polish category is a tuple $\mathcal{C} = (\mathcal{C}_{\mathsf{Obj}}, \mathcal{C}_{\mathsf{Mor}}, \sigma, \tau, \iota, \circ)$ s.t.:

- $\mathcal{C}_{\mathsf{Obj}}$ (objects) and $\mathcal{C}_{\mathsf{Mor}}$ (morphisms) are quasi-Polish spaces.
- $\sigma \colon \mathcal{C}_{\mathsf{Mor}} \to \mathcal{C}_{\mathsf{Obj}}$ (source), $\tau \colon \mathcal{C}_{\mathsf{Mor}} \to \mathcal{C}_{\mathsf{Obj}}$ (target), and $\iota \colon \mathcal{C}_{\mathsf{Obj}} \to \mathcal{C}_{\mathsf{Mor}}$ (identity) are continuous functions.
- $\circ :\subseteq \mathcal{C}_{\mathsf{Mor}} \times \mathcal{C}_{\mathsf{Mor}} \to \mathcal{C}_{\mathsf{Mor}}$ (composition) is a partial continuous function whose domain is all pairs $\langle g, f \rangle$ of morphisms with $\sigma(g) = \tau(f)$.
- The usual axioms of a category hold ($\circ$ is associative, etc.).

$\mathcal{C}$ is an effective quasi-Polish category if $\mathcal{C}_{\mathsf{Obj}}$ and $\mathcal{C}_{\mathsf{Mor}}$ are effective quasi-Polish spaces and σ, τ, ι, and $\circ$ are computable functions.

Computable functors

Internal functors are also defined in the usual way. Let $\mathcal{C}$ and $\mathcal{D}$ be quasi-Polish categories.

Definition

A continuous (computable) functor $F\colon \mathcal{C} \to \mathcal{D}$ is a pair $F = (F_{\mathsf{Obj}}, F_{\mathsf{Mor}})$ of continuous (computable) functions $F_{\mathsf{Obj}}\colon \mathcal{C}_{\mathsf{Obj}} \to \mathcal{D}_{\mathsf{Obj}}$ and $F_{\mathsf{Mor}}\colon \mathcal{C}_{\mathsf{Mor}} \to \mathcal{D}_{\mathsf{Mor}}$ satisfying

- $F_{\mathsf{Obj}} \circ \sigma_{\mathcal{C}} = \sigma_{\mathcal{D}} \circ F_{\mathsf{Mor}}$,
- $F_{\mathsf{Obj}} \circ \tau_{\mathcal{C}} = \tau_{\mathcal{D}} \circ F_{\mathsf{Mor}}$,
- $F_{\mathsf{Mor}} \circ \iota_{\mathcal{C}} = \iota_{\mathcal{D}} \circ F_{\mathsf{Obj}}$, and
- $F_{\mathsf{Mor}}(g \circ_{\mathcal{C}} f) = F_{\mathsf{Mor}}(g) \circ_{\mathcal{D}} F_{\mathsf{Mor}}(f)$ for all composable $f, g \in \mathcal{C}_{\mathsf{Mor}}$.

(i.e., F respects source, target, identity, and composition).

Computable contravariant functors (which switch the direction of the morphisms) are defined similarly.

Computable natural transformations

Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be continuous (computable) functors.

Definition

A continuous (computable) natural transformation from $\eta\colon F \to G$ is given by a continuous (computable) function $\hat{\eta}\colon \mathcal{C}_{\mathsf{Obj}} \to \mathcal{D}_{\mathsf{Mor}}$ satisfying:

- $\sigma_{\mathcal{D}} \circ \hat{\eta} = F_{\mathsf{Obj}}$ and $\tau_{\mathcal{D}} \circ \hat{\eta} = G_{\mathsf{Obj}}$ (i.e., $\hat{\eta}(C)\colon F_{\mathsf{Obj}}(C) \to G_{\mathsf{Obj}}(C)$ for each $C \in \mathsf{Obj}_{\mathcal{C}}$),
- $\hat{\eta}(\tau_{\mathcal{C}}(f)) \circ_{\mathcal{D}} F_{\mathsf{Mor}}(f) = G_{\mathsf{Mor}}(f) \circ_{\mathcal{D}} \hat{\eta}(\sigma_{\mathcal{C}}(f))$ for each $f \in \mathcal{C}_{\mathsf{Mor}}$.

$$\begin{tikzcd} F_{\mathsf{Obj}}(C) \arrow[r, "F_{\mathsf{Mor}}(f)"] \arrow[d, "\hat{\eta}(C)"'] & F_{\mathsf{Obj}}(D) \arrow[d, "\hat{\eta}(D)"] \\ G_{\mathsf{Obj}}(C) \arrow[r, "G_{\mathsf{Mor}}(f)"] & G_{\mathsf{Obj}}(D) \end{tikzcd} \qquad (f\colon C \to D \text{ in } \mathcal{C})$$

Functor categories

- Although we will not use them in this talk, for many applications (e.g., constructing effective categories of sheaves) it is useful to introduce functor categories.
- Given categories $\mathcal{C}$ and $\mathcal{D}$, the functor category $\mathcal{D}^{\mathcal{C}}$ has functors from $\mathcal{C}$ to $\mathcal{D}$ as objects and natural transformations as morphisms:
  - $(\mathcal{D}^{\mathcal{C}})_{\mathsf{Obj}}$ is the subspace of $\mathcal{D}_{\mathsf{Obj}}^{\mathcal{C}_{\mathsf{Obj}}} \times \mathcal{D}_{\mathsf{Mor}}^{\mathcal{C}_{\mathsf{Mor}}}$ of functors.
  - $(\mathcal{D}^{\mathcal{C}})_{\mathsf{Mor}}$ is the subspace of $\mathcal{D}_{\mathsf{Mor}}^{\mathcal{C}_{\mathsf{Mor}}} \times (\mathcal{D}^{\mathcal{C}})_{\mathsf{Obj}} \times (\mathcal{D}^{\mathcal{C}})_{\mathsf{Obj}}$ of tuples $\langle \eta, F, G \rangle$ such that η is a natural transformation from F to G.

Theorem

If $\mathcal{C}$ and $\mathcal{D}$ are quasi-Polish categories and $\mathcal{C}_{\mathsf{Mor}}$ is locally compact, then $\mathcal{D}^{\mathcal{C}}$ is a quasi-Polish category.

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Discrete spaces and Hausdorff spaces

Let $\mathbb{S} = \{\top, \bot\}$ be the Sierpinski space (as a space of ideals, $\mathbb{S} \cong \mathbf{I}(\prec_{\mathbb{S}})$, where $x \prec_{\mathbb{S}} y$ iff $x, y \in \{0, 1\}$ and $x \leq y$).

Definition

Let $X \cong \mathbf{I}(\prec)$ be an effective quasi-Polish space. X is

- computably discrete if $(=)\colon X \times X \to \mathbb{S}$ is computable.
  - i.e., there is a c.e. enumeration $(\langle n_i, m_i \rangle)_{i\in\mathbb{N}}$ such that $I = J$ in $\mathbf{I}(\prec)$ iff there exists $i \in \mathbb{N}$ with $n_i \in I$ and $m_i \in J$.
- computably Hausdorff if $(\neq)\colon X \times X \to \mathbb{S}$ is computable.
  - i.e., there is a c.e. enumeration $(\langle n_i, m_i \rangle)_{i\in\mathbb{N}}$ such that $I \neq J$ in $\mathbf{I}(\prec)$ iff there exists $i \in \mathbb{N}$ with $n_i \in I$ and $m_i \in J$.

- $2^{\mathbb{N}}$ is computably Hausdorff but not computably discrete.
- Unsolvability of the word problem for groups yields examples of spaces (finitely presented groups) that are computably discrete but not computably Hausdorff.

Overt spaces and Compact spaces

Definition

Let $X \cong \mathbf{I}(\prec)$ be an effective quasi-Polish space. X is

- computably overt if $(\exists_X)\colon \mathbb{S}^X \to \mathbb{S}$ $(\exists_X(U) = \bot \iff U = \emptyset)$ is computable.
  - i.e., $\{a \in \mathbb{N} \mid [a]_\prec \neq \emptyset\}$ is c.e. Equivalently, the maximal element $X \in \mathbf{A}(X)$ is computable.
- computably compact if $(\forall_X)\colon \mathbb{S}^X \to \mathbb{S}$ $(\forall_X(U) = \top \iff U = X)$ is computable.
  - i.e., X is classically compact and there is a c.e. enumeration of all finite covers of X by basic opens. Equivalently, the minimal element $X \in \mathbf{K}(X)$ exists and is computable.

- $\mathbb{N}$ is computably overt but not computably compact.
- If $A \subseteq \mathbb{N}$ is Π^0_1 but not Σ^0_1, then $X = \{0^\infty\} \cup \{0^n 1^\infty \mid n \in A\} \subseteq 2^{\mathbb{N}}$ is computably compact but not computably overt.

Overt discrete quasi-Polish spaces (ODS)

A partial equivalence relation (PER) is a symmetric transitive relation.

Definition (ODS)

- $\mathsf{ODS}_{\mathsf{Obj}} = \{\equiv \in \mathbf{A}(\mathbb{N}\times\mathbb{N}) \mid \ \equiv \text{ is a PER}\}$.
- $\mathsf{ODS}_{\mathsf{Mor}} \subseteq \mathbf{A}(\mathbb{N}\times\mathbb{N}) \times \mathsf{ODS}_{\mathsf{Obj}} \times \mathsf{ODS}_{\mathsf{Obj}}$ is the subspace of all tuples $\langle G, \equiv_\sigma, \equiv_\tau\rangle$ satisfying (for $a, a', b, b' \in \mathbb{N}$):
  1. $G(a,b)$ implies $a \equiv_\sigma a \,\&\, b \equiv_\tau b$.
  2. $G(a,b) \,\&\, a \equiv_\sigma a'$ implies $G(a',b)$.
  3. $G(a,b) \,\&\, b \equiv_\tau b'$ implies $G(a,b')$.
  4. $G(a,b) \,\&\, G(a,b')$ implies $b \equiv_\tau b'$.
  5. $a \equiv_\sigma a$ implies $(\exists b \in \mathbb{N})\, G(a,b)$.
- $\sigma(\langle G, \equiv_\sigma, \equiv_\tau\rangle) = \equiv_\sigma$, $\tau(\langle G, \equiv_\sigma, \equiv_\tau\rangle) = \equiv_\tau$, and $\iota(\equiv) = \langle \equiv, \equiv, \equiv\rangle$.
- $\langle G, \equiv_\rho, \equiv_\tau\rangle \circ \langle F, \equiv_\sigma, \equiv_\rho\rangle = \langle (G\circ F), \equiv_\sigma, \equiv_\tau\rangle$, where $(G\circ F)(a,c) \iff (\exists b \in \mathbb{N})[F(a,b) \,\&\, G(b,c)]$.

Overt discrete quasi-Polish spaces (ODS)

- Each $\equiv$ determines a setoid $S_\equiv = (\{a \in \mathbb{N} \mid a \equiv a\}, \equiv)$.
  - Alternatively, we can think of $\equiv$ as determining a space of equivalence classes $\{[a]_\equiv \mid a \in \mathbb{N} \,\&\, a \equiv a\} \subseteq \mathbf{A}(\mathbb{N})$.
- The lower Vietoris topology provides **positive** information about the setoid and the graph of functions:
  - $a \in S_\equiv$ if and only if $\equiv \in \Diamond\{\langle a, a\rangle\}$.
  - $f(a) = b$ if and only if $Graph(f) \in \Diamond\{\langle a, b\rangle\}$.
  - On the other hand, $a \notin S_\equiv$ and $f(a) \neq b$ are not always open.
- PERs have a long history in constructive mathematics and computability theory. The computable points of $\mathsf{ODS}_{\mathsf{Obj}}$ are equivalent to the **computably discrete computably quasi-Polish spaces** recently studied by E. Neumann, A. Pauly, C. Pradic, & M. Valenti, which they showed are equivalent (when non-empty) to the **computably enumerable equivalence relations (ceer)** studied by S. Gao & P. Gerdes.
  - $X \leq_m Y$ iff there is a computable mono $f\colon X \to Y$.

Overt discrete quasi-Polish spaces (ODS)

Theorem

- ODS is an effective quasi-Polish category.
- ODS is equivalent to the category of overt discrete quasi-Polish spaces. Under this equivalence:
  - The computable points of $\mathsf{ODS}_{\mathsf{Obj}}$ are the computably overt computably discrete effective quasi-Polish spaces.
  - The computable points of $\mathsf{ODS}_{\mathsf{Mor}}$ are the computable functions between computably overt computably discrete effective quasi-Polish spaces.
- Finite products and equalizers exist and are computable from the relevant diagrams.

- A computable point of X is a computable function $p\colon 1 \to X$, where 1 is the terminal object (i.e., the singleton space $\mathbf{I}(\equiv_1)$ where $m \equiv_1 n$ for all $m, n \in \mathbb{N}$).
- Equivalently, $I \in \mathbf{I}(\prec)$ is a computable point if $I \subseteq \mathbb{N}$ is c.e.

Compact Hausdorff quasi-Polish spaces (CHS)

Definition (CHS)

- $\mathsf{CHS}_{\mathsf{Obj}} = \{\equiv \in \mathbf{K}(2^{\mathbb{N}} \times 2^{\mathbb{N}}) \mid \equiv \text{ is a PER}\}$.
- $\mathsf{CHS}_{\mathsf{Mor}} \subseteq \mathbf{K}(2^{\mathbb{N}} \times 2^{\mathbb{N}}) \times \mathsf{CHS}_{\mathsf{Obj}} \times \mathsf{CHS}_{\mathsf{Obj}}$ is the subspace of all tuples $\langle G, \equiv_\sigma, \equiv_\tau \rangle$ satisfying (for $x, x', y, y' \in 2^{\mathbb{N}}$):
  1. $G(x, y)$ implies $x \equiv_\sigma x \,\&\, y \equiv_\tau y$
  2. $G(x, y) \,\&\, x \equiv_\sigma x'$ implies $G(x', y)$
  3. $G(x, y) \,\&\, y \equiv_\tau y'$ implies $G(x, y')$
  4. $G(x, y) \,\&\, G(x, y')$ implies $y \equiv_\tau y'$
  5. $x \equiv_\sigma x$ implies $(\exists y \in 2^{\mathbb{N}})\, G(x, y)$.
- $\sigma(\langle G, \equiv_\sigma, \equiv_\tau \rangle) = \equiv_\sigma$, $\tau(\langle G, \equiv_\sigma, \equiv_\tau \rangle) = \equiv_\tau$, and $\iota(\equiv) = \langle \equiv, \equiv, \equiv \rangle$.
- $\langle G, \equiv_\rho, \equiv_\tau \rangle \circ \langle F, \equiv_\sigma, \equiv_\rho \rangle = \langle (G \circ F), \equiv_\sigma, \equiv_\tau \rangle$, where $(G \circ F)(x, z) \iff (\exists y \in 2^{\mathbb{N}})[F(x, y) \,\&\, G(y, z)]$.

Compact Hausdorff quasi-Polish spaces (CHS)

- Each $\equiv$ determines a setoid $S_\equiv = (\{x \in 2^{\mathbb{N}} \mid x \equiv x\}, \equiv)$.
  - Alternatively, we can think of $\equiv$ as determining a space of equivalence classes $\{[x]_\equiv \mid x \in 2^{\mathbb{N}} \,\&\, x \equiv x\} \subseteq \mathbf{K}(2^{\mathbb{N}})$.
- The upper Vietoris topology provides **negative** information about the setoid and the graph of functions:
  - $x \notin S_\equiv$ if and only if $\equiv \in \Box((2^{\mathbb{N}} \times 2^{\mathbb{N}}) \setminus \{\langle x, x\rangle\})$.
  - $f(x) \neq y$ if and only if $Graph(f) \in \Box((2^{\mathbb{N}} \times 2^{\mathbb{N}}) \setminus \{\langle x, y\rangle\})$.
  - On the other hand, $x \in S_\equiv$ and $f(x) = y$ are not always open.
- So when checking that CHS is well-defined, it is more natural to look at the contrapositives of the statements.
  - For example, composition is well-defined and continuous (even computable) because

$$\neg(G \circ F)(x, z) \iff (\forall y \in 2^{\mathbb{N}})[\neg F(x, y) \vee \neg G(y, z)],$$

    and the right hand side is open because it is universal quantification of an open predicate over a compact space.

Compact Hausdorff quasi-Polish spaces (CHS)

Theorem

- CHS is an effective quasi-Polish category.
- CHS is equivalent to the category of compact Hausdorff quasi-Polish spaces. Under this equivalence:
  - The computable points of $\mathsf{CHS}_{\mathsf{Obj}}$ are the computably compact computably Hausdorff effective quasi-Polish spaces.
  - The computable points of $\mathsf{CHS}_{\mathsf{Mor}}$ are the computable functions between computably compact computably Hausdorff effective quasi-Polish spaces.
- Finite products and equalizers exist and are computable from the relevant diagrams.

Classically, every compact Hausdorff quasi-Polish space is Polish, hence CHS is equivalent to the category of compact Polish spaces. However, the computable objects of CHS are not necessarily overt, which is required for "computable Polish spaces".

Dual contravariant functors

Definition

Define the contravariant functor $\mathcal{D}\colon \mathsf{ODS} \to \mathsf{CHS}$ as follows:

- For $\equiv$ in $\mathsf{ODS}_{\mathsf{Obj}}$, define $\mathcal{D}_{\mathsf{Obj}}(\equiv)$ to be the object $\equiv'$ in $\mathsf{CHS}_{\mathsf{Obj}}$ defined as (for $x, y \in 2^{\mathbb{N}}$):

$$x \equiv' y \iff (\forall a, b \in \mathbb{N})[a \equiv b \Rightarrow x(a) = y(b)].$$

- For $\langle G, \equiv_\sigma, \equiv_\tau \rangle$ in $\mathsf{ODS}_{\mathsf{Mor}}$, define $\mathcal{D}_{\mathsf{Mor}}(\langle G, \equiv_\sigma, \equiv_\tau \rangle)$ to be the morphism $\langle G', \equiv'_\tau, \equiv'_\sigma \rangle$ in $\mathsf{CHS}_{\mathsf{Mor}}$ defined as:
  - $\equiv'_\sigma = \mathcal{D}_{\mathsf{Obj}}(\equiv_\sigma)$ and $\equiv'_\tau = \mathcal{D}_{\mathsf{Obj}}(\equiv_\tau)$
  - $G' \in \mathbf{K}(2^{\mathbb{N}} \times 2^{\mathbb{N}})$ is defined so that $G'(y, x)$ holds if and only if

$$y \equiv'_\tau y \,\&\, x \equiv'_\sigma x \,\&\, (\forall a, b \in \mathbb{N})[G(a, b) \Rightarrow x(a) = y(b)].$$

Intuitively, $\mathcal{D}$ maps X to the space 2^X of continuous functions from X to the discrete space $2 = \{0, 1\}$, and maps $f\colon X \to Y$ to $2^f : 2^Y \to 2^X$ defined as $2^f = \lambda g : 2^Y.\lambda x : X.g(f(x))$.

Dual contravariant functors

Fix a c.e. enumeration of all finite subsets of $2^{<\mathbb{N}}$ (the set of finite binary sequences), and for each $a \in \mathbb{N}$ define $\Phi_a \colon 2^{\mathbb{N}} \to 2$ as $\Phi_a(x) = 1$ if and only the a-th finite subset of $2^{<\mathbb{N}}$ contains a prefix of x (thus $(\Phi_a)_{a\in\mathbb{N}}$ is an enumeration of all continuous functions from $2^{\mathbb{N}}$ to 2).

Definition

Define the contravariant functor $\mathcal{E} \colon \mathsf{CHS} \to \mathsf{ODS}$ as follows:

- For $\equiv$ in $\mathsf{CHS}_{\mathsf{Obj}}$, define $\mathcal{E}_{\mathsf{Obj}}(\equiv)$ to be the object $\equiv'$ in $\mathsf{ODS}_{\mathsf{Obj}}$ defined as (for $a, b \in \mathbb{N}$):

$$a \equiv' b \iff (\forall x, y \in 2^{\mathbb{N}})[x \equiv y \Rightarrow \Phi_a(x) = \Phi_b(y)].$$

- For $\langle G, \equiv_\sigma, \equiv_\tau \rangle$ in $\mathsf{CHS}_{\mathsf{Mor}}$, define $\mathcal{E}_{\mathsf{Mor}}(\langle G, \equiv_\sigma, \equiv_\tau \rangle)$ to be the morphism $\langle G', \equiv'_\tau, \equiv'_\sigma \rangle$ in $\mathsf{ODS}_{\mathsf{Mor}}$ defined as:
    - $\equiv'_\sigma = \mathcal{E}_{\mathsf{Obj}}(\equiv_\sigma)$ and $\equiv'_\tau = \mathcal{E}_{\mathsf{Obj}}(\equiv_\tau)$
    - $G' \in \mathbf{A}(\mathbb{N} \times \mathbb{N})$ is defined so that $G'(b, a)$ holds if and only if

$$b \equiv'_\tau b \,\&\, a \equiv'_\sigma a \,\&\, (\forall x, y \in 2^{\mathbb{N}})[G(x, y) \Rightarrow \Phi_a(x) = \Phi_b(y)].$$

Dual contravariant functors

Definition

Natural transformations $\eta\colon 1 \to \mathcal{E} \circ \mathcal{D}$ and $\epsilon\colon 1 \to \mathcal{D} \circ \mathcal{E}$ are defined as follows.

1. $\hat{\eta}\colon \mathsf{ODS}_{\mathsf{Obj}} \to \mathsf{ODS}_{\mathsf{Mor}}$ is defined as $\hat{\eta}(\equiv) = \langle G_\eta, \equiv, \equiv'' \rangle$, where
   - $\equiv' = \mathcal{D}(\equiv)$ and $\equiv'' = \mathcal{E}(\equiv')$
   - $G_\eta(a, b)$ if and only if

$$a \equiv a \,\&\, b \equiv'' b \,\&\, (\forall x \in 2^{\mathbb{N}})[x \equiv' x \Rightarrow \Phi_b(x) = x(a)].$$

2. $\hat{\epsilon}\colon \mathsf{CHS}_{\mathsf{Obj}} \to \mathsf{CHS}_{\mathsf{Mor}}$ is defined as $\hat{\epsilon}(\equiv) = \langle G_\epsilon, \equiv, \equiv'' \rangle$, where
   - $\equiv' = \mathcal{E}(\equiv)$ and $\equiv'' = \mathcal{D}(\equiv')$
   - $G_\epsilon(x, y)$ if and only if

$$x \equiv x \,\&\, y \equiv'' y \,\&\, (\forall a \in \mathbb{N})[a \equiv' a \Rightarrow y(a) = \Phi_a(x)].$$

Intuitively, $\eta_X : X \to 2^{2^X}$ is $\eta_X(x) = \lambda f : 2^X . f(x)$, and similarly for ϵ.

Dual contravariant functors

Lemma

$\mathcal{D}\colon \mathsf{ODS} \to \mathsf{CHS}$ and $\mathcal{E}\colon \mathsf{CHS} \to \mathsf{ODS}$ are computable contravariant functors, and $\eta\colon 1 \to \mathcal{E} \circ \mathcal{D}$ and $\epsilon\colon 1 \to \mathcal{D} \circ \mathcal{E}$ are computable natural transformations.

Theorem

- $\mathcal{D}(\eta_X) \circ \epsilon_{\mathcal{D}(X)} = 1_{\mathcal{D}(X)}$ and $\mathcal{E}(\epsilon_Y) \circ \eta_{\mathcal{E}(Y)} = 1_{\mathcal{E}(Y)}$ hold for each $X \in \mathsf{ODS}_{\mathsf{Obj}}$ and $Y \in \mathsf{CHS}_{\mathsf{Obj}}$.
- Therefore, $\mathcal{D}$ and $\mathcal{E}$ are adjoint on the right, meaning there is a natural bijection between morphisms of the form $f\colon X \to \mathcal{E}(Y)$ in ODS and $g\colon Y \to \mathcal{D}(X)$ in CHS given by $f = \mathcal{E}_{\mathsf{Mor}}(g) \circ \hat{\eta}(X)$ and $g = \mathcal{D}_{\mathsf{Mor}}(f) \circ \hat{\epsilon}(Y)$.

Intuitively, given $f\colon X \to 2^Y$, the corresponding $g\colon Y \to 2^X$ is $g = \lambda y.\lambda x.f(x)(y)$ (the double transpose).

Category of Boolean algebras

Definition

Given a quasi-Polish category $\mathcal{C}$ with finite products, we define $\mathsf{BA}(\mathcal{C})$ to be the category of Boolean algebra objects in $\mathcal{C}$.

- $\mathsf{BA}(\mathcal{C})_{\mathsf{Obj}}$ is the subspace of $\mathcal{C}_{\mathsf{Obj}} \times (\mathcal{C}_{\mathsf{Mor}})^5$ of tuples $(A, \vee, \wedge, \neg, 0, 1)$, where
  - $\vee, \wedge \colon A \times A \to A$
  - $\neg \colon A \to A$
  - $0, 1 \colon 1_{\mathcal{C}} \to A$

  satisfy the Boolean algebra axioms (associativity, distributivity, commutativity, absorption, idempotency, identity, complement).
- $\mathsf{BA}(\mathcal{C})_{\mathsf{Mor}}$ is the subspace of $\mathcal{C}_{\mathsf{Mor}} \times \mathsf{BA}(\mathcal{C})_{\mathsf{Obj}} \times \mathsf{BA}(\mathcal{C})_{\mathsf{Obj}}$ of all triples $(h, (A, \vee, \wedge, \neg, 0, 1), (B, \vee', \wedge', \neg', 0', 1'))$ such that $h \colon A \to B$ is a Boolean algebra homomorphism.
- Composition and identity morphisms are inherited from $\mathcal{C}$.

Category of Boolean algebras

- The forgetful functor $U\colon \mathsf{BA}(\mathcal{C}) \to \mathcal{C}$ (which just omits the Boolean algebra structure) is computable.
- If $\mathcal{C}$ is an effective quasi-Polish category and products are computable (i.e., $A \times A$ can be computed from A, etc.) then $\mathsf{BA}(\mathcal{C})$ is an effective quasi-Polish category.
- In particular, $\mathsf{BA}(\mathsf{ODS})$ and $\mathsf{BA}(\mathsf{CHS})$ are effective quasi-Polish categories.
  - $\mathsf{BA}(\mathsf{ODS})$ is the category of countable models of the Π_2-theory of Boolean algebras in the sense of R. Chen's work.
    - Inequality ($\neq$) is not assumed to be a basic formula.
    - R. Chen uses the corresponding groupoid in his applications, which restricts the space of morphisms to isomorphisms.

Zero-dimensional compact Hausdorff spaces (ZCHS)

Definition

$Y \in \mathsf{CHS}_{\mathsf{Obj}}$ is 2-sober if the following is an equalizer:

$$Y \xrightarrow{\epsilon_Y} \mathcal{DE}(Y) \underset{\epsilon_{\mathcal{DE}(Y)}}{\overset{\mathcal{DE}(\epsilon_Y)}{\rightrightarrows}} \mathcal{DEDE}(Y)$$

Lemma

Let Y be a (computable) object of CHS. The following are equivalent:

(i) Y is 2-sober.

(ii) For all $x, y \in 2^{\mathbb{N}}$,

$$x \equiv y \iff (\forall a, b \in \mathbb{N})[a \equiv' b \Rightarrow \Phi_a(x) = \Phi_b(y)],$$

where $\equiv$ and $\equiv'$ are PERs corresponding to Y and $\mathcal{E}(Y)$.

(iii) The space corresponding to Y is computably homeomorphic to a closed (Π^0_1) subspace of $2^{\mathbb{N}}$.

Zero-dimensional compact Hausdorff spaces (ZCHS)

Definition

Define ZCHS to be the full subcategory of CHS of 2-sober objects.

Lemma

ZCHS is an effective quasi-Polish category.

- For every $Y \in \mathsf{CHS}_{\mathsf{Obj}}$, there is a unique morphism from Y to the equalizer of the 2-sober diagram, and the morphism can be computed uniformly in Y. Since the subspace of $\mathsf{CHS}_{\mathsf{Mor}}$ of isomorphisms is Π^0_2, it follows that $\mathsf{ZCHS}_{\mathsf{Obj}} \subseteq \mathsf{CHS}_{\mathsf{Obj}}$ is Π^0_2.

Computable Stone duality

Theorem

The categories BA(ODS) and ZCHS are computably dually equivalent. This means that there exists:

- computable contravariant functors $\mathsf{pt}\colon \mathsf{BA(ODS)} \to \mathsf{ZCHS}$ and $\mathsf{clop}\colon \mathsf{ZCHS} \to \mathsf{BA(ODS)}$,
- computable natural isomorphisms $\eta\colon 1_{\mathsf{BA(ODS)}} \to \mathsf{clop} \circ \mathsf{pt}$ and $\epsilon\colon 1_{\mathsf{ZCHS}} \to \mathsf{pt} \circ \mathsf{clop}$.

- $\mathsf{pt}_{\mathsf{Obj}}((A, \vee, \wedge, \neg, 0, 1))$ is the Π^0_1 subspace of $\mathcal{D}_{\mathsf{Obj}}(A)$ of Boolean algebra homomorphisms $h\colon A \to 2$.
- $\mathsf{pt}_{\mathsf{Mor}}(f)$ is the restriction of $\mathcal{D}_{\mathsf{Mor}}(f)$.
- clop is basically $\mathcal{E}$, but with Boolean algebra structure added.
- η and ϵ are basically restrictions of the natural transformations defined earlier.
- The inverses of η and ϵ are obtained by flipping their graphs (hence computable), but proving that these are valid morphisms is non-constructive.

Computable Stone duality (dual version)

Theorem

Dually, the categories BA(ZCHS) and ODS are computably dually equivalent.

The diagram below summarizes the situation:

$$
\begin{tikzcd}[column sep=large, row sep=large]
\mathsf{ODS} \arrow[r, shift right=1ex, "2^{(-)}"'] \arrow[d, shift right=1ex, "2^{2^{(-)}}"'] \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\dashv"] & \mathsf{BA(ZCHS)} \arrow[l, shift right=1ex, "\mathsf{pt}"'] \arrow[d, shift left=1ex, "U"] \arrow[d, phantom, "\dashv"] \\
\mathsf{BA(ODS)} \arrow[u, shift right=1ex, "U"'] \arrow[r, shift left=1ex, "\mathsf{pt}"] \arrow[r, phantom, "\cong"] & \mathsf{ZCHS} \arrow[l, shift left=1ex, "2^{(-)}"] \arrow[u, shift left=1ex, "2^{2^{(-)}}"]
\end{tikzcd}
$$

- $\mathcal{D}$, $\mathcal{E}$ are written as $2^{(-)}$,
- η, ϵ are written as $2^{2^{(-)}}$,
- U is the forgetful functor,
- the horizontal functors are contravariant (the dual equivalences).

Conclusion

- We constructed ODS and CHS as effective quasi-Polish categories, and showed that full Stone duality between BA(ODS) and ZCHS, and between BA(ZCHS) and ODS, are computable in the TTE sense.
- Future work:
  - Investigate subcategories (e.g., overt Hausdorff discrete spaces and overt Hausdorff compact spaces).
  - Investigate supercategories such as locally compact Hausdorff spaces.
  - Investigate other dualities (e.g., Pontryagin duality - see work by A. Melnikov)