On Computability of Universal Block-Diagonalization via Finite Group Representations

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Outline

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For every ***real symmetric*** *matrix A there exists an orthogonal matrix U such that $U^T AU$ is diagonal.*

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Fact (DFT)

*There exists a unitary matrix U of order n such that $U^* AU$ is diagonal for every **circulant** matrix A of order n.*

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Fact

There exists an ***orthogonal*** *matrix U of order n such that $U^T AU$ is block-diagonal for every real* ***centrosymmetric*** *matrix A of order n.*

Note: $\exists U \ \forall A$

Universal Block-Diagonalizer (for real matrices)

Definition

*Let $\mathcal{A}$ be a class of $n \times n$ **real** matrices. Let $k \in \mathbb{N}$ and $d_1, \ldots, d_k \in \mathbb{N}$ such that $d_1 + \cdots + d_k = n$. A matrix $U \in O(n)$ is an **universal block diagonalizer** for $\mathcal{A}$ if $U^T A U$ is block-diagonal with blocks of size $d_1, \ldots, d_k$ for every $A \in \mathcal{A}$ and U does not depend on entries of A.*

Example

Definition (centrosymmetric matrix)

An $n \times n$ matrix A is **centrosymmetric** if $AJ = JA$ where J is the *reverse-identity* matrix $J = \mathrm{antidiag}(1, \dots, 1)$.

Example (UBD for even-order centrosymmetric)

An $2n \times 2n$ matrix A is centrosymmetric iff there exist $n \times n$ matrices B, C such that

$$A = \begin{pmatrix} B & CJ \\ JC & JBJ \end{pmatrix}$$

It is well known that the algebra of $2n \times 2n$ centrosymmetric matrices has a universal block diagonalizer

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ J & -J \end{pmatrix}.$$

Note: a similar example exists for centrosymmetric matrices of odd order.

TTE: represented spaces

Definition

*A **representation** of a set X is a partial surjection*

$$\delta_X :\subseteq \mathbb{N}^{\mathbb{N}} \to X.$$

*If $\delta_X(p) = x$, then p is a δ_X-**name** of x. The pair (X, δ_X) is a represented space.*

Definition

*For represented spaces (X, δ_X) and (Y, δ_Y), a map $f :\subseteq X \to Y$ is **computable** if some Type-2 machine computes a realizer $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that*

$$\delta_Y(F(p)) = f(\delta_X(p))$$

for all $p \in \mathrm{dom}(f \circ \delta_X)$.

Multi-valued maps and Cauchy names

Linear-algebra tasks can have many valid answers, so we use multi-valued maps

$$f :\subseteq X \rightrightarrows Y.$$

A Type-2 machine computes f if, from every name of $x \in \mathrm{dom}(f)$, it outputs a name of some $y \in f(x)$.

Definition (Cauchy representation)

A ρ-name of $x \in \mathbb{R}$ is a sequence of rationals $(q_m)_{m\in\mathbb{N}}$ with

$$|q_m - x| \leq 2^{-m} \qquad \forall m \in \mathbb{N}.$$

Matrices in $M_n(\mathbb{R})$ are named by interleaving the names of their individual entries i.e. representation $\rho^{n\times n}$. Finite data, such as n or a Cayley table, are given exactly as finite strings.

Question

Is the map

$$A \mapsto U_A$$

computable?

Proposition (M. Ziegler and V. Brattka [3])

The multi-valued eigenbasis problem

$$\mathrm{Sym}_n \rightrightarrows O(n), \qquad A \mapsto \{U : U^{-1}AU \textit{ is diagonal}\}$$

*is **not** Type-2 computable.*

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The advised multi-valued problem

$$(A, |\sigma(A)|) \mapsto \{U : U^{-1}AU \text{ is diagonal}\}$$

is Type-2 computable.

Expectations

- Computing the universal block diagonalizer for a given semisimple algebra is in general not Type-2 computable
- if we provide some discrete advice then it might be computable

Main Question

Expectation: For a given matrix $A \in \mathcal{A}$, if the block sizes $d_1, \ldots, d_k \in \mathbb{N}$ of A are given as input, then a block-diagonalizer of A is computable

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Given a matrix class $\mathcal{A}$, can we computably obtain the universal block diagonalizer for the whole class?

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We focus on the case when $\mathcal{A}$ is a *semisimple* algebra of matrices.

Definition (Wedderburn-Artin)

A subalgebra $\mathcal{A} \subseteq M_n(\mathbb{R})$ is ***semisimple*** *if*

$$\mathcal{A} \cong \bigoplus_i M_{m_i}(D_i), \quad D_i \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$$

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Definition

The representation ρ is *irreducible* if $\mathbb{R}^n$ has no ρ-invariant subspace other than $\{0\}$ and $\mathbb{R}^n$ itself.

Note: By $\rho_1, \ldots, \rho_k$ we denote the pairwise non-isomorphic irreducible representations of G over $\mathbb{R}$.

Definition

The *character* of ρ is the function $\chi_\rho\colon G \to \mathbb{R}$ defined by

$$\chi_\rho(g) = \operatorname{tr}(\rho(g))$$

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Fact

Every representation ρ decomposes as

$$\rho \cong m_1\rho_1 \oplus \cdots \oplus m_k\rho_k,$$

where $m_i \in \mathbb{N}$ is the multiplicity of ρ_i in ρ.

Definition

The *i-th isotypic subspace* of ρ is $V_i = \mathrm{im}(P_i)$, where

$$P_i = \frac{n_i}{\delta_i|G|}\sum_{g\in G}\chi_i(g)\,\rho(g)$$

is the orthogonal projection onto V_i; where n_i is the degree of ρ_i and δ_i is the Frobenius-Schur indicator.

Definition

Let G be a finite group. Let $\rho : G \to \mathrm{GL}_n(\mathbb{R})$ be a representation of G. Then the set

$$\mathcal{C}(G,\rho) = \{A \in M_n(\mathbb{R}) \mid A\rho(g) = \rho(g)A, \ \textit{for all } g \in G\}$$

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Note (Maschke)

A commutant $\mathcal{C}(G,\rho)$ of a finite group representation is semisimple.

Let G be given by its Cayley table and let $\rho : G \to GL_n(\mathbb{R})$ be given by Cauchy names for the matrices $\rho(g)$.

Theorem (UBD for finite group commutant is computable)

Let G be a finite group. Let $\rho : G \to \mathrm{O}(n)$ be a representation of G. Then the map $\mathrm{FinGrp} \times \mathrm{O}(n)^{\mathrm{FinGrp}} \rightrightarrows O_n(\mathbb{R})$ given by

$$(G, \rho) \mapsto U$$

*where $U^T AU$ is block-diagonal for each $A \in \mathcal{C}(G, \rho)$, is **uniformly** Type-2 computable.*

Proof (sketch)

On input (G, ρ), for each $i = 1, \dots, k(G)$:

1. compute the irreducible character χ_i
2. compute the projector

$$P_i = \frac{n_i}{\delta_i |G|} \sum_{g \in G} \chi_i(g^{-1}) \rho(g)$$

with $\delta_i = 2^{1-\nu_i}$ where ν_i are the Frobenius-Schur indicators

$$\nu_i = \frac{1}{|G|} \sum_{g \in G} \chi_i(g^2) \in \{1, 0, -1\}$$

3. compute $d_i = n_i \cdot m_i$ where $n_i = \chi_i(e)$ and $m_i = tr(P_i)/n_i$ by approximating $tr(P_i)$ from the Cauchy names of $\rho(g)$ and finding a rational $\tilde{m}_i$, $|m_i - \tilde{m}_i| < 1/2$ and rounding to the nearest integer
4. compute the orthonormal basis $\mathcal{B}_i$ of $V_i = \mathrm{Im}\ P_i$ using Gram-Schmidt orthogonalization, the dimension of V_i is now known to be exactly d_i

Return $U = [\mathcal{B}_1 | \dots | \mathcal{B}_{k(G)}]$

Example (centrosymmetric matrices) (1/5)

The group is $\mathbb{Z}/2\mathbb{Z} = \{e, s\}$ with the Cayley table:

$\cdot$	e	s
e	e	s
s	s	e

Let $n = 4$. Let $\rho\colon \mathbb{Z}/2\mathbb{Z} \to \mathrm{GL}_4(\mathbb{R})$,

$$\rho(e) = I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \rho(s) = J_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Example (centrosymmetric matrices) (2/5)

The group $\mathbb{Z}/2\mathbb{Z}$ has two conjugacy classes $\{e\}$ and $\{s\}$, and two irreducible representations, both of degree 1:

	e	s
χ_1 (trivial)	1	1
χ_2 (sign)	1	-1

The degrees are $n_1 = \chi_1(e) = 1$ and $n_2 = \chi_2(e) = 1$.

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The degrees are $n_1 = \chi_1(e) = 1$ and $n_2 = \chi_2(e) = 1$. For each $\rho(g)$, the algorithm approximates the traces:

$$\chi_\rho(e) = \mathrm{tr}(I_4) = 4, \qquad \chi_\rho(s) = \mathrm{tr}(J_4) = 0.$$

Example (centrosymmetric matrices) (3/5)

Using the formula $m_i = \frac{1}{|G|}\sum_{g\in G}\chi_\rho(g)\,\chi_i(g^{-1})$ with $|G| = 2$:

Trivial (χ_1):

$$m_1 = \frac{1}{2}(\chi_\rho(e)\cdot\chi_1(e) + \chi_\rho(s)\cdot\chi_1(s)) = \frac{1}{2}(4\cdot 1 + 0\cdot 1) = 2.$$

Sign (χ_2):

$$m_2 = \frac{1}{2}(\chi_\rho(e)\cdot\chi_2(e) + \chi_\rho(s)\cdot\chi_2(s)) = \frac{1}{2}(4\cdot 1 + 0\cdot(-1)) = 2.$$

In the Type-2 setting, these sums are only approximated, but once the error is less than $1/2$, rounding recovers the exact integers: $m_1 = 2$, $m_2 = 2$.

Example (centrosymmetric matrices) (4/5)

Using $P_i = \frac{n_i}{|G|} \sum_{g \in G} \chi_i(g)\, \rho(g)$:

Trivial projection ($n_1 = 1$, $\chi_1(e) = 1$, $\chi_1(s) = 1$):

$$P_1 = \frac{1}{2}(\rho(e) + \rho(s)) = \frac{1}{2}(I_4 + J_4) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

$\mathrm{Im}\, P_1 = \{x \in \mathbb{R}^4 : J_4 x = x\}$, confirming $d_1 = \mathrm{rank}(P_1) = 2$.

Sign projection ($n_2 = 1$, $\chi_2(e) = 1$, $\chi_2(s) = -1$):

$$P_2 = \frac{1}{2}(\rho(e) - \rho(s)) = \frac{1}{2}(I_4 - J_4) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

$\mathrm{Im}\, P_2 = \{x \in \mathbb{R}^4 : J_4 x = -x\}$, confirming $d_2 = \mathrm{rank}(P_2) = 2$.

Example (centrosymmetric matrices) (5/5)

The universal block-diagonalizer is

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

Corollary

Let G be a finite group. Let $\rho : G \to \mathrm{O}(n)$ be a representation of G. Then the map $\rho^{\mathrm{n}\times\mathrm{n}} \times \mathrm{FinGrp} \times \mathrm{O}(n)^{\mathrm{FinGrp}} \rightrightarrows O_n(\mathbb{R})$ given by

$$(A, G, \rho) \mapsto U$$

where $U^T AU$ is block-diagonal for each $A \in \mathcal{C}(G, \rho)$, is uniformly Type-2 computable.

Question

Given a nonsingular matrix $A \in \mathcal{C}(G, \rho)$ can we calculate the QR factorization of A such that $Q, R \in \mathcal{C}(G, \rho)$? In other words, is

$$A \mapsto (Q, R)$$

Type-2 computable?

Corollary

Suppose ρ is of real type i.e. all corresponding $\delta_i = 1$. Then map $\rho^{n\times n} \rightrightarrows \rho^{n\times n} \times \rho^{n\times n}$,

$$A \mapsto (Q, R)$$

such that

1. $A, Q, R \in \mathcal{C}(G, \rho)$,
2. *A is nonsingular,*
3. $A = QR$
4. *Q is orthogonal and*
5. *R is block-upper-triangular*

is Type-2 computable.

Some Remarks

Note

The theorem gives the isotypic decomposition. The full Wedderburn-Artin decomposition is future work.

Open Questions

Question (generalizing to other groups)

Can we generalize this result for commutants $\mathcal{C}(G, \rho)$ where G is:

- *a finitely generated group ?*
- *a compact group (e.g. can we apply Peter-Weyl theory for compact groups)?*
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Some well-known classes of matrices e.g symmetric or Toeplitz are not commutants. For what other classes of matrices can we obtain similar results about computability of UBD?

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Question (Weihrauch degrees)

In general, the UBD is not computable for semisimple algebras. What are the Weihrauch degrees in such cases?

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Thank you!