Computable Homeomorphisms from One-Point Metric Bases

Konrad Burnik * Zvonko Iljazović **

*kburnik@gmail.com

**University of Zagreb, Croatia

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Outline

Definition

A triple (X, d, α) is a **computable metric space** if (X, d) is a metric space and $\alpha : \mathbb{N} \to X$ is a sequence with a dense image in X such that the function $\mathbb{N}^2 \to \mathbb{R}$

$$(i, j) \mapsto d(\alpha_i, \alpha_j)$$

is computable. We call the points $\alpha_0, \alpha_1, \ldots$ **rational points** or **special points**.

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Definition

- A set I is a **rational ball** if $I = B(\lambda, \rho)$ where λ is a rational point and $\rho \in \mathbb{Q}^+$.
- We denote by (I_k) and $(\widehat{I_k})$ some fixed effective enumerations of open and closed rational balls, respectively.
- A finite union of open rational balls is called **rational open set**.
- Denote by (J_j) some fixed effective enumeration of rational open sets.

Definition

A *point* $x \in X$ is **computable** in (X, d, α) if there is a computable function $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that

$$d(x, \alpha_{f(k)}) < 2^{-k}$$

for all $k \in \mathbb{N}$.

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Definition

A *sequence* $x : \mathbb{N} \to X$ is **computable** in (X, d, α) if there is a computable function $f : \mathbb{N}^2 \longrightarrow \mathbb{N}$ such that

$$d(x_n, \alpha_{f(n,k)}) < 2^{-k}$$

for all $n, k \in \mathbb{N}$.

Definition

Let (X, d, α) be a computable metric space. Let $S \subseteq X$ be closed. We say that S is a **co-computably enumerable** in (X, d, α) if there exists a c.e. $\Omega \subseteq \mathbb{N}$ such that

$$X \setminus S = \bigcup_{i \in \Omega} I_i$$

Definition

A computable metric space (X, d, α) is **effectively compact** if (X, d) is complete and it is *effectively totally bounded*, i.e. there is a computable function $\varphi : \mathbb{N} \to \mathbb{N}$ such that for each $k \in \mathbb{N}$

$$X = \bigcup_{i=0}^{\varphi(k)} B(\alpha_i, 2^{-k}).$$

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Definition

Let (X, d, α) be a computable metric space. We say that (X, d, α) has **effective covering property** if

$$\{(i,j) \in \mathbb{N}^2 \mid \widehat{I}_i \subseteq J_j\}$$

is c.e.

Definition

Let (X, d, α) and (Y, d, β) be computable metric spaces. We say that $f : X \to Y$ is **computable** if

1. f maps computable sequences to computable sequences
2. f is *effectively σ-uniformly continuous* i.e.
   There exists a computable function $\delta : \mathbb{N}^2 \to \mathbb{N}$ and $a \in X$ such that $d(x, y) < 2^{-\delta(M,k)}$ implies $d(f(x), f(y)) < 2^{-k}$ for all $M, k \in \mathbb{N}$ and all $x, y \in B(a, M)$.

We say that f is a **computable homeomorphism** if f is a homeomorphism and f and f^{-1} are computable.

Fact

Every connected 1-manifold (with or without boundary) is homeomorphic to either $[0, 1]$, $\mathbb{S}^1$, $[0, \infty)$ *or* $\mathbb{R}$.

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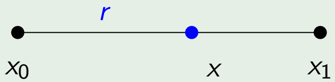
Question

Suppose (X, d, α) *is a computable metric space that is a connected 1-manifold (with or without boundary). Suppose* $X \sim S$ *where* S *is exactly one of* $[0,1]$, $\mathbb{S}^1$, $[0,\infty)$ *or* $\mathbb{R}$.

Can we then also find a ***computable*** *homeomorphism* $S \to X$*?*

Example

Consider a segment in $\mathbb{R}$ with endpoints x_0 and x_1.



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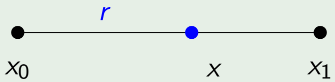

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Note

It can be shown that the same property holds for the endpoint x_1.

Definition (Melter & Tomescu, 1984)

Let (X, d) be a metric space. A non-empty subset $S \subseteq X$ is called a resolving set if

$$\forall x, y \in X,\ d(x, s) = d(y, s) \text{ for all } s \in S \implies x = y.$$

A resolving set of minimal cardinality is called a metric basis for (X, d).

In other words: the distances to the points of S determine every point of X uniquely.

Example

- in $(\mathbb{R}^n, d_2)$ every finite set of $n+1$ geometrically independent points is a metric basis
- in (I^2, d_2), the set $\{(0,0),(1,0)\}$ is a metric basis
- in (I^2, d_∞), the set $\{(0,0),(0,1),(1,0)\}$ is a metric basis
- in $(\mathbb{R}^2, d_\infty)$, does not have a finite metric basis (see Melter & Tomescu, 1984)
- any dense subset of a metric space (X,d) is a resolving set

based on joint work with Lucija Validžić

Proposition

If (X, d) is a compact and connected metric space which has a one-point metric basis then (X, d) is an arc.

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Let (X, d) be an arc. If x_0 is a metric basis for (X, d) then x_0 is an endpoint.

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Theorem

Assume that (X, d, α) is an ***effectively compact*** *computable metric space such that the space (X, d) has* ***finitely many connected components****. If $x_0 \in X$ is metric basis for (X, d), then x_0 is a computable point in (X, d, α).*

Proof outline for the Theorem

- Show that $\{x_0\}$ is *co-computably enumerable* in (X, d, α) and use a Lemma presented on CCA 2024
- From effective compactness of (X, d, α) it is not hard to show that $\{x_0\}$ co-computably enumerable implies that x_0 is a computable point in (X, d, α).

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Example

Let $\gamma > 0$ be left-computable but not computable real. Consider computable metric space constructed from a segment in $\mathbb{R}$ with endpoints 0 and γ.

Note that this space is not effectively compact. Now γ is a metric basis of this space, but γ is not computable.

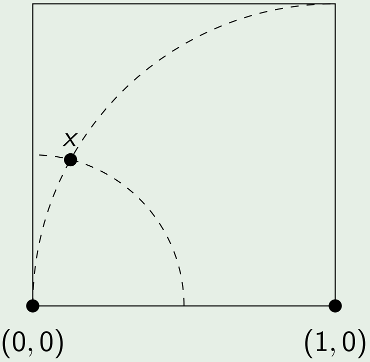
Example

Note

Further, the claim of the Theorem does not hold for metric bases which contain more than one point.

Example

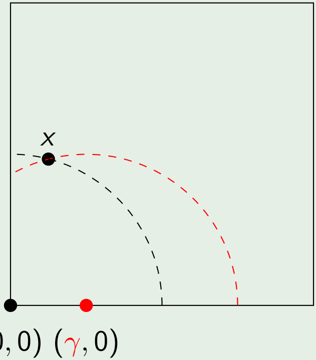
Consider I, the unit square $[0,1]^2$ with euclidean metric.



Here, $\{(0,0),(1,0)\}$ is a two-point metric basis for I and both of its points are computable.

Example

On the other hand, for $\gamma \in \langle 0, 1 \rangle$ uncomputable, we also have the following.



Here, $\{(0,0), (\gamma, 0)\}$ is a two-point metric basis for I, which contains a non-computable point.

Main Result (Compact Case)

Theorem

Let (X, d, α) be a computable metric space that is effectively compact and such that (X, d) is connected. Suppose (X, d) has a one-point metric basis. Then there exists a computable homeomorphism $[0, 1] \to X$.

Proof (sketch)

1. By assumption $\{x_0\}$ is a one-point metric basis for (X, d) that according to Theorem must be computable in (X, d, α).

   Let $h : X \to [0, +\infty)$,

   $$h(x) = d(x_0, x), \quad \forall x \in X.$$

   Due to compactness of X, the codomain of h can be restricted to $[0, t]$ for some $t > 0$.

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   Due to compactness of X, the codomain of h can be restricted to $[0, t]$ for some $t > 0$.

2. Let $f : [0, +\infty) \to X$,

   $$f(r) \text{ is the unique element of } X \text{ such that } d(f(r), x_0) = r$$

   Due to compactness of X, the domain of f can be restricted to $[0, t]$ for some $t > 0$.

3. Prove: existence and uniqueness of such x for each $r \in [0, t]$

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2. Prove: continuity of f
3. Prove: computability of f (follows from effective compactness)
4. Prove: $h_*^{-1} = f_*$ (where $h_* : [0,t] \to X$ and $f_* : X \to [0,t]$ are given by $h_*(r) = h(r)$ for all $r \in [0,t]$ and $f_*(x) = f(x)$ for all $x \in X$.
5. Let $g : [0,1] \to [0,t]$ be any computable homeomorphism. Then $h_* \circ g : [0,1] \to X$ is a computable homeomorphism.

Proposition

Let (X, d) be a connected metric space which is not compact. If (X, d) has a one-point metric basis then (X, d) is a topological ray.

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Theorem

*Let (X, d, α) be a computable metric space which is a **topological ray** and that has the **effective covering property** and **compact closed balls**. If $x_0 \in X$ is a metric basis for (X, d) then x_0 is a computable point in (X, d, α).*

Definition (Isometry on a subset)

Let (X, d_X) and (Y, d_Y) be metric spaces.

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$$d_Y(f(x), f(y)) = d_X(x, y)$$

for all $x \in X$ and for all $y \in S$.

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Example

Let (X, d) be a connected non-compact metric space and $\{x_0\}$ a metric basis for (X, d). Let $f : [0, +\infty) \to X$,

$$f(r) \text{ is the unique element of } X \text{ such that } d(f(r), x_0) = r$$

for all $r \in [0, +\infty)$. Then $x_0 = f(0)$ and f is an isometry w.r.t. $\{0\}$.

Main Result (Non-compact Case)

Theorem

*Let (X, d, α) be a computable metric space with the **effective covering property** and compact closed balls. Suppose (X, d) is **non-compact**, **connected** and has a one-point metric basis. Then there exists a computable homeomorphism $[0, +\infty\rangle \to X$.*

Proof (sketch)

1. By assumption $\{x_0\}$ is a one-point metric basis for (X, d) and $f(0) = x_0$. According to Theorem, x_0 must be computable in (X, d, α).

   Let $h : X \to [0, +\infty)$,

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   for all $r \in [0, +\infty)$.
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3. Prove: f is computable (both computable sequence mapping and eff.-σ uniform continuity is obtained from covering X with effective chains that have additional c.e. properties, the fact that links of the chain cover the correct segment of the image of f follows from f being an isometry w.r.t. $\{0\}$)

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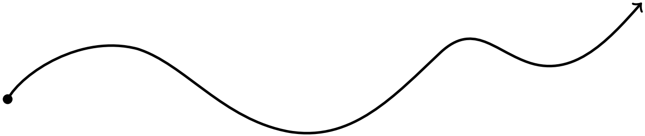
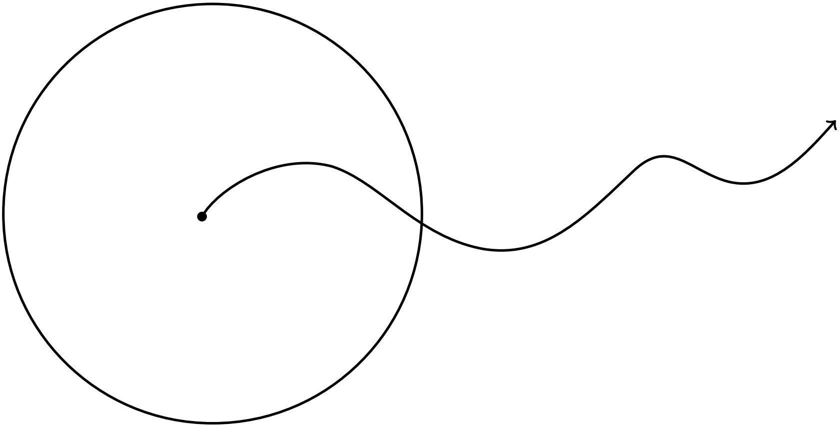
4. Prove: $h^{-1} = f$.

Conclusion: f is a computable homeomorphism $[0, +\infty) \to X$.

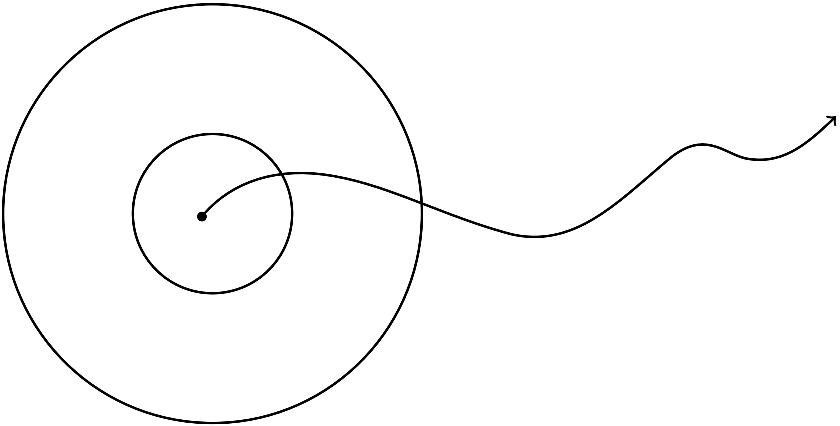
Proof sketch: effective σ-uniform continuity of f (1/7)



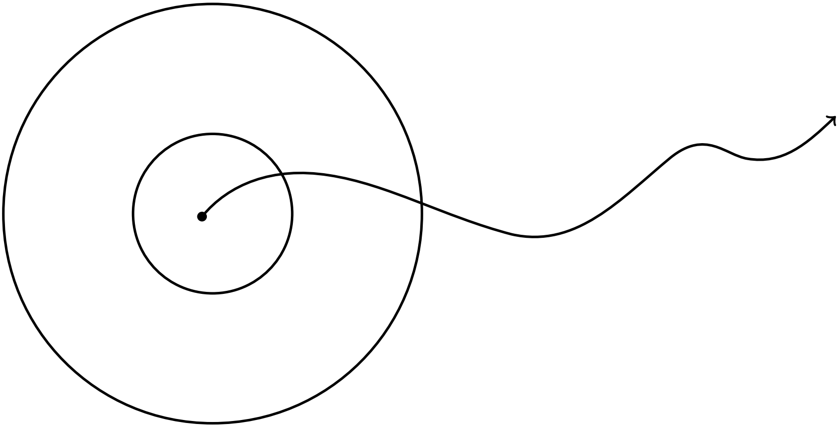
Proof sketch: effective σ-uniform continuity of f (2/7)



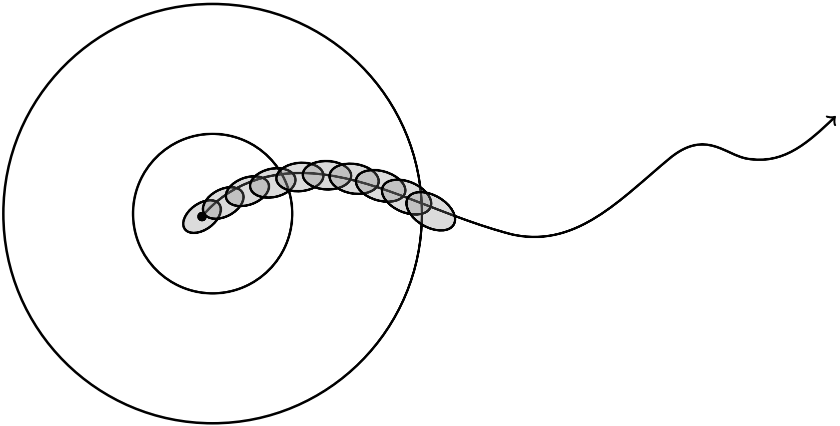
Proof sketch: effective σ-uniform continuity of f (3/7)



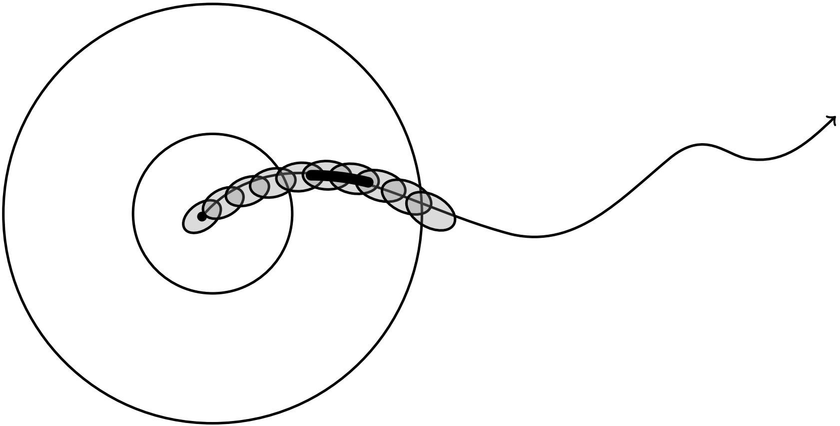
Proof sketch: effective σ-uniform continuity of f (4/7)



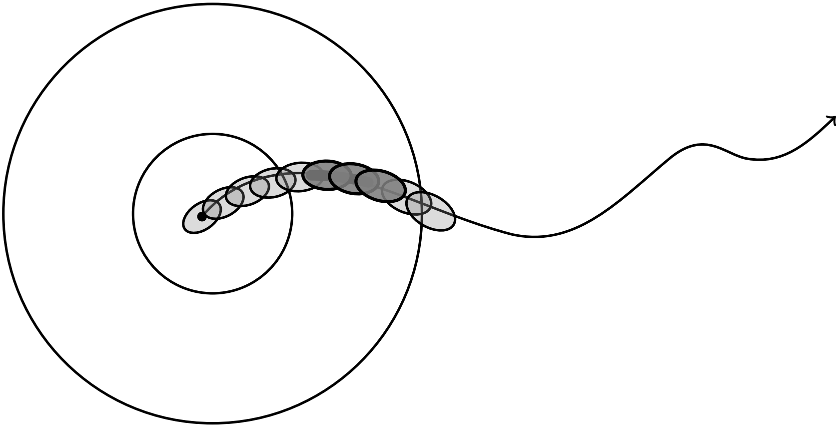
Proof sketch: effective σ-uniform continuity of f (5/7)



Proof sketch: effective σ-uniform continuity of f (6/7)



Proof sketch: effective σ-uniform continuity of f (6/7)



Question

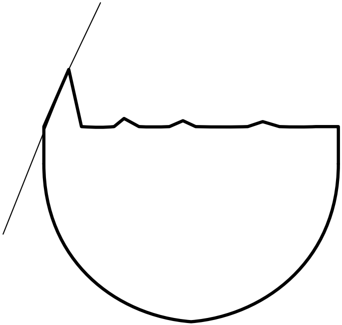
Can computable homeomorphisms $S \to X$ where S is either $\mathbb{S}^1$ or $\mathbb{R}$ also be obtained from one-point metric bases in a similar way?

Fact

A topological circle or line never has a one-point metric basis: at least two points are always required.

Counterexample

There exists a topological circle X with a computable metric basis S, $|S| = 2$, such that *no* computable homeomorphism $\mathbb{S}^1 \to X$ exists.



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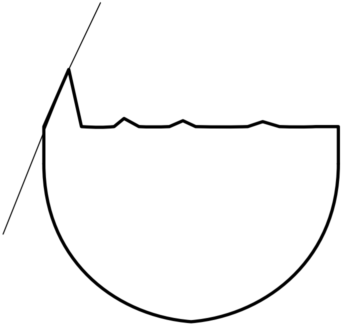

Note

A similar example can be constructed for the topological line.

What is missing?

Additional assumptions are required, e.g. *effective local connectedness*, as mentioned in Miller (2002)

References (1/2)

Konrad Burnik, Zvonko Iljazović.
Computability of 1-manifolds.
Logical Methods in Computer Science, Vol. 10(2:8):1–28, 2014.

Zvonko Iljazović.
Local computability of computable metric spaces and computability of co-c.e. continua.
Glasnik Matematički, 47(1):1-20, 2012.

Zvonko Iljazović and Lucija Validžić.
Maximal computability structures.
Bulletin of Symbolic Logic, 22(4):445–468, 2016.

References (2/2)

Alexander Melnikov.
Computably isometric spaces
Journal of Symbolic Logic, 78:1055–1085, 2013.

Marian Pour-El and Ian Richards.
Computability in Analysis and Physics.
Springer-Verlag, Berlin-Heielberg-New York, 1989.

M. Yasugi, T. Mori and Y. Tsujji.
Effective properties of sets and functions in metric spaces with computability structure.
Theoretical Computer Science, 219:467–486, 1999.

M. Yasugi, T. Mori and Y. Tsujji.
Computability structures on metric spaces.
Combinatorics, Complexity and Logic
Proc. DMTCS96 (D.S. Bridges et al), Springer, Berlin, 351–362, 1996.

Thank you!