A new notion of reducibility for third-order arithmetic

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RUB Bochum, Department of Philosophy II

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In a nutshell

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Theorem A is computationally as strong as Theorem B,

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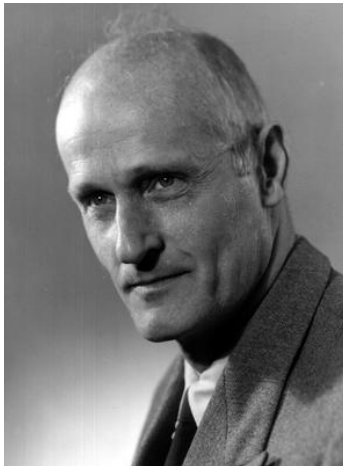
A lot of these results are based on jww Dag Normann (Oslo).

Turing and Kleene



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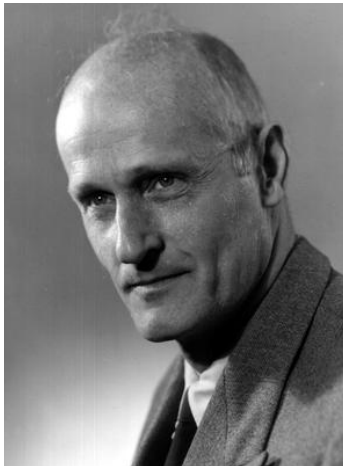

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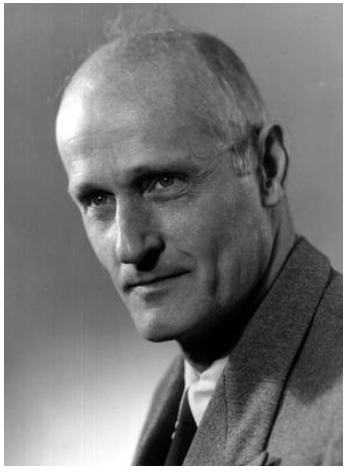

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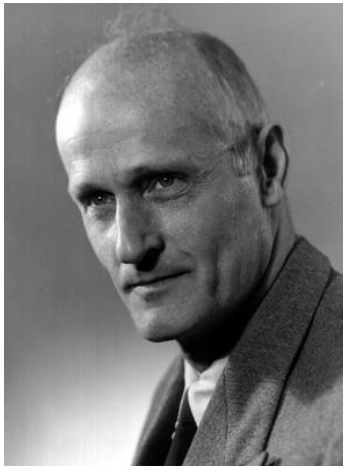

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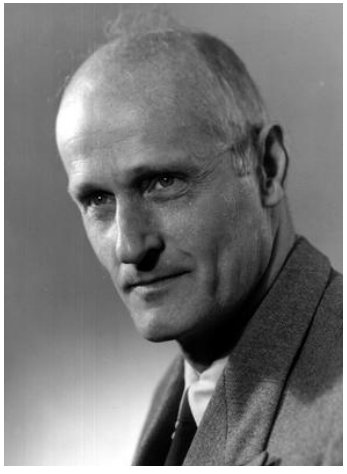

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Infinite time Turing machine (ITTM): running time measured by ordinals (Hamkins, Lewis); higher-order inputs possible (Welsh)

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The first and second point are due to Normann; the third point is due to Normann-Sanders.

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while higher-order computability theory studies whether:

$$(\forall \Theta^3)\big[(\forall Y^2)A(Y, \Theta(Y)) \to (\forall Z^2)B(Z, \{e\}(\Theta, Z))\big] \qquad \text{(K)}$$

where $\{e\}(\Theta, Z)$ is the e-th Kleene S1-S9 algorithm.

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A version of (K) involving Turing machines:

$$(\forall Z^2,x^1)\big[A(t(Z),x) \to [\{e_0\}^{s(x,Z)}\downarrow \wedge\, B(Z,\{e_0\}^{s(x,Z)})]\big], \qquad \text{(N)}$$

where $t^{2\to 2}, s^{(1\times 2)\to 1}$ are primitive recursive and $\{e_0\}^{s(x,Z)}$ is a Turing machine with oracle $s(x,Z)$.

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- Consequent of (K) then reduces to Turing computability.

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- Sometimes: 'μN-reduction' where we allow $t(Z, \exists^2)$.

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Suprema, jumps, and comprehension

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- Convergence theorems for nets indexed by countable sets.

Suprema, jumps, and comprehension

The following (strongly) μN-reduce to each other:

- For Y^2, the set $\{n \in \mathbb{N} : (\exists x \in \mathbb{N}^{\mathbb{N}})(Y(x, n) = 0)\}$ exists.
- (Darboux, 1872) For $f : [0, 1] \to [0, 1]$, $\sup_{x \in [0,1]} f(x)$ exists.
- Convergence theorems for nets indexed by $\mathbb{N}^{\mathbb{N}}$ (Moore-Smith).
- ...

The following (strongly) μN-reduce to each other:

- A countable (=injection to $\mathbb{N}$) subset of $[0, 1]$ has an enumeration.
- For $f \in BV$, $\sup_{x \in [0,1]} f(x)$ exists.
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Functions of bounded variation (BV) were introduced in 1881 by Jordan when studying the convergence of Fourier series.

Non-classical

Grilliot's trick

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Grilliot's trick = a method for obtaining Turing jump functional E (aka Kleene's $\exists^2$) from a discontinuous function on $\mathbb{R}$ or $\mathbb{N}^{\mathbb{N}}$.

$$(\forall f : \mathbb{N} \to \mathbb{N})(E(f) = 0 \leftrightarrow (\exists n \in \mathbb{N})(f(n) = 0)). \qquad (\exists^2)$$

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There are many equivalences for $(\exists^2)$ in HORM (see ...).

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We have no idea how to tackle the S1-S9 versions of these conjectures.

Recall N-reduction

Solving the problem $(\forall Z^2)(\exists y^1)B(Z,x)$ N-reduces to solving the problem $(\forall Y^2)(\exists x^1)A(Y,x)$.

in case:

$$(\forall Z^2, x^1)\big[A(t(Z),x) \to [\{e\}^{s(x,Z)}\downarrow \wedge\, B(Z,\{e\}^{s(x,Z)})]\big], \qquad \text{(N)}$$

where t, s are primitive recursive and $\{e\}^y$ is a TM with oracle y^1.

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Consider the Heine-Borel theorem/Cousin's lemma (HBU):

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For $\Psi : [0,1] \to \mathbb{R}^+$ and the 'canonical' covering $\cup_{x\in[0,1]} B(x, \Psi(x))$, there are $x_0, \ldots, x_k \in [0,1]$ such that $\cup_{i\leq k} B(x_i, \Psi(x_i))$ still covers $[0,1]$, i.e. finite sub-covering.

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Conjecture: NIN does not N-reduce to HBU for Baire class 2.

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Origins of N-reduction

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Definition (NFP)

For any formula $A(n^0)$, we have

$$(\forall f^1)(\exists n^0)A(\overline{f}n) \to (\exists \gamma \in K_0)(\forall f^1)A(\overline{f}\gamma(f)), \tag{1}$$

where '$\gamma \in K_0$' means that γ^1 is a (total) Kleene associate.

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If NFP is restricted to A with type 2 parameters (only), then we can study it using N-reductions.

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Makoto Fujiwara: explicit definition restricted to primitive recursive reductions. Different goal: capture constructive reasoning via classical means.

Final Thoughts

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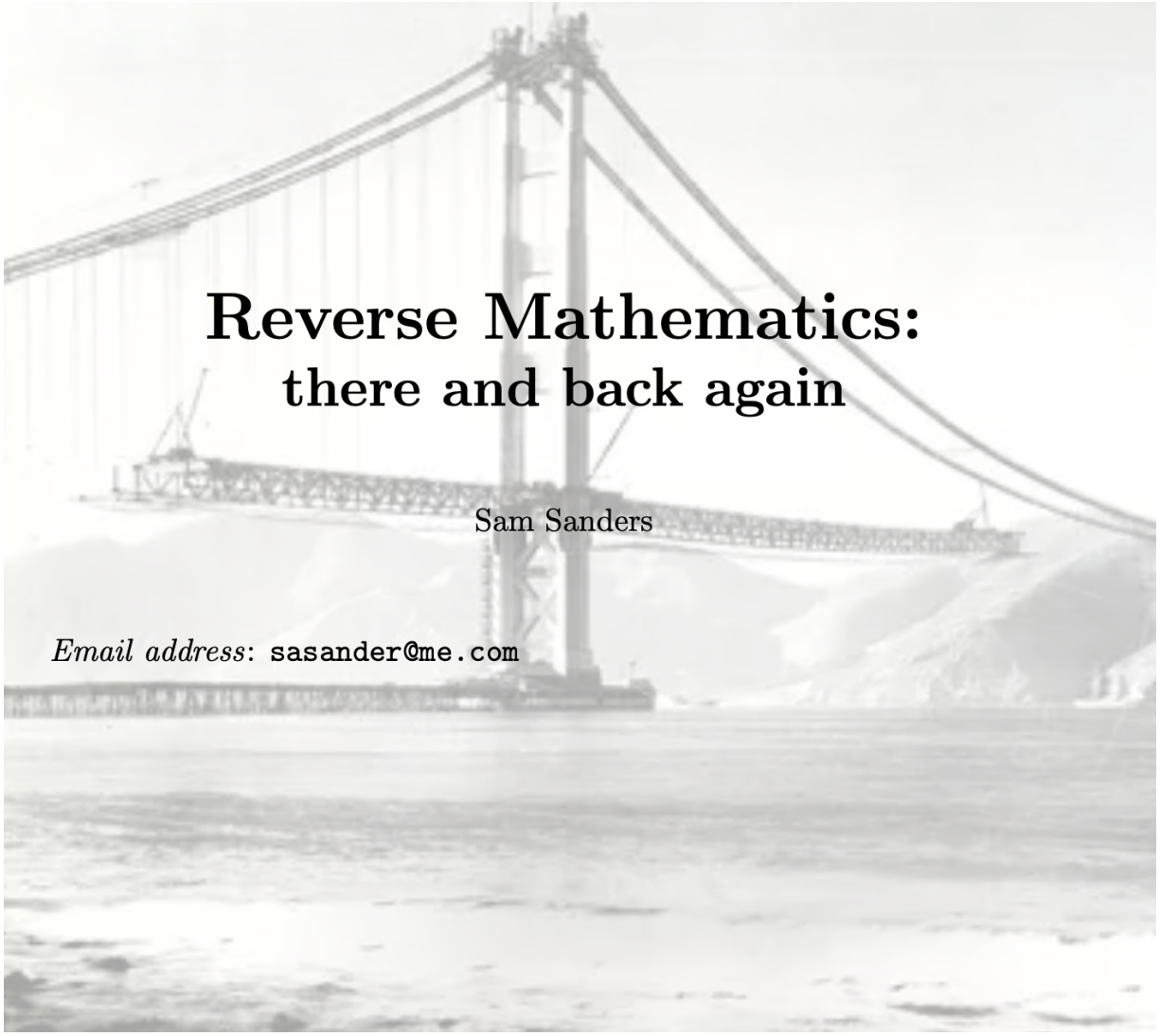
Any questions?

And thanks to AN for all valuable advise, without which N-reduction would not have been discovered/invented.

Book on Reverse Mathematics



The details of N-reduction are in one of the final chapters.

Higher types redux-ish

Higher-order computability theory studies whether:

$$(\forall \Theta^3)\big[(\forall Y^2)A(Y, \Theta(Y)) \to (\forall Z^2)B(Z, \{e\}(\Theta, Z))\big] \qquad \text{(K)}$$

where $\{e\}(\Theta, Z)$ is a Kleene S1-S9 algorithm.

A version of (K) involving Turing machines:

$$(\forall Z^2, x^1)\big[A(t(Z), x) \to [\{e_0\}^{s(x,Z)}\downarrow \wedge B(Z, \{e_0\}^{s(x,Z)})]\big], \qquad \text{(N)}$$

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In case (N), we say that

solving the problem $(\forall Z^2)(\exists y^1)B(Z,x)$ *N*-reduces to solving the problem $(\forall Y^2)(\exists x^1)A(Y,x)$.