An effective Banach-Mazur game and an application to dynamical systems

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In Memoriam

Vladimir V'yugin
(1948-2024)

Main results

- A definition of effectively meager sets.
- Two characterizations of effective meager sets using Banach-Mazur games.
- An effective Poincaré recurrence theorem for effective category
- Other applications

Definitions

Topological Space

(X, τ) is said to be a topological space if X is a non-empty set and $\tau \subseteq \mathcal{P}(X)$ is a collection of sets satisfying the following:

- $X, \varnothing \in \tau$.
- τ is closed under arbitrary unions.
- τ is closed under finite intersections.

Basis

For a topological space (X, τ), a family of sets $\beta \subseteq \tau$ is called a basis if for every $x \in X$ and neighborhood $V \in \tau$ of x, there is $U \in \beta$ such that $x \in U \subset V$.

Let $\Sigma = \{0, 1\}$.

Effective representations

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Representation

For a countable class $\mathcal{U}$, the partial computable surjection $\nu : \Sigma^* \dashrightarrow \mathcal{U}$, where $(\forall w \in dom(\nu))(\nu(w) \neq \varnothing)$, is said to be a representation of $\mathcal{U}$.

For $U \in \mathcal{U}$, $\nu^{-1}(U)$ refers to a name of U.

Effective T_0 spaces

Computable T_0 Space

A computable T_0 space is a tuple (X, τ, β, ν) such that

- (X, τ) is a second countable T_0 space,
- β is a countable basis
- $\nu : \Sigma^* \dashrightarrow \beta$ is a representation of a countable basis β

where

$$\forall u, v \in dom(\nu), \nu(u) \cap \nu(v) = \bigcup\{\nu(w) : (u, v, w) \in B\}$$

for some c.e. set B, and $U \neq \varnothing$ for $U \in \beta$.

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for some c.e. set B, and $U \neq \varnothing$ for $U \in \beta$.

(X, τ, β, ν) is said to be a strongly computable T_0 space if $(\forall u, v \in dom(\nu))\ \nu(u) \cap \nu(v)$ is decidable.

Effectively meager set

C.E. Open Set

A set U is said to be a computably enumerable open (c.e. open) set if it is a computably enumerable union of the basic open sets.

A set F is said to be a co-c.e. closed set if F^c is c.e. open.

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Effective Nowhere Dense Set

Let (X, τ, β, ν) be a computable T_0 space. A set $A \subseteq X$ is said to be effectively nowhere dense in X if there exists a computable function $f : \Sigma^* \to \Sigma^*$ such that for $w \in dom(\nu)$, $\nu(f(w)) \subseteq (\nu(w) \setminus A)^o$.

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Effective First Category Set

A set is said to be of effective first category, if it can be represented as a c.e. union of effective nowhere dense sets.

Results

Lemma 1

Let (X, τ, β, ν) be a computable T_0 space. Then the following hold.

- Any finite union of effective nowhere dense sets is effectively nowhere dense.
- The closure of an effective nowhere dense set is effectively nowhere dense.

Lemma 2

- The complement of a dense c.e. open set is effective nowhere dense.
- The complement of an effective nowhere dense set contains a dense c.e. open set.

Banach-Mazur Game

$BM\langle A, B\rangle$

- Let I_0 be a closed interval.

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  - Player 2 chooses a closed interval $I_2 \subseteq I_1$.

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  - Player 1 chooses a closed interval $I_1 \subseteq I_0$.
  - Player 2 chooses a closed interval $I_2 \subseteq I_1$.
  - Player 1 chooses a closed interval $I_3 \subseteq I_2 \ldots$
- If $A \cap \bigcap_n I_n \neq \varnothing$, player 1 wins. Else, player 2 wins.

Effectivizing the Game

- assumptions:
  - (X, τ, β, ν) is a strongly computable T_0 space.
  - All the playable sets lie in the class $\mathcal{G}$ of sets with non-empty interior and such that every non-empty open set of the space contains some set from $\mathcal{G}$.
  - Player 2 has a c.e. strategy.

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Effective Strategy for Player 2

An effective strategy for the second player is denoted by

$$\mathcal{G}^{(2)} = \left\langle G_{2k} : G_{2k} = \nu(f_k\left(\nu^{-1}(G_1), \cdots, \nu^{-1}(G_{2k-1})\right))\right\rangle_{k \geq 1} \subseteq \mathcal{G},$$

where $\{f_n : f_n : (\Sigma^*)^{2n-1} \to \Sigma^*\}_{n \geq 1}$ is a uniformly computable (in n) sequence of computable choice functions, where, for each $i \in \mathbb{N}$, $G_i \in \mathcal{G}^{(2)}$.

Theorem

Theorem (Koul, Nandakumar 26)

In a strongly computable T_0 space (X, τ, β, ν) with $M \coprod C = X$, the game $BM\langle M, C\rangle$ has an effective winning strategy for P_2 if and only if M is an effective first category set in X.

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Proof:

- Let M be an effective first category set.

  Then, $M = \bigcup_{n\geq 1} M_n$, where each M_n is an effective nowhere dense set.
- At stage k, if P_1 plays the set $G_{2k-1} \in \mathcal{G}$, P_1 responds with G_{2k}, the first set in the enumeration of $G_{2k-1} \setminus \overline{M_k}$.

Theorem

$$
\begin{aligned}
M \bigcap \left(\bigcap_{n \geq 1} G_n \right) &= \bigcap_{k \geq 1} G_{2k-1} \cap G_{2k} \\
&\subseteq \bigcap_{k \geq 1} G_{2k-1} \cap (G_{2k-1} \setminus \overline{M_k}) \\
&= \bigcap_{k \geq 1} G_{2k-1} \setminus \bigcup_{k \geq 1} \overline{M_k} \\
&\subseteq M \bigcap \left(\bigcap_{n \geq 1} G_n \cap \overline{M_n}^c \right) \\
&\subseteq M \bigcap \left(\bigcap_{n \geq 1} G_n \cap M_n^c \right) \\
&= \varnothing.
\end{aligned}
$$

Thus P_2 has a winning strategy.

Converse

Conversely, let P_2 have an effective winning strategy $G_1, G_2, G_3, \ldots,$ where $G_{2k} = f_k\left(G_1, G_2, \cdots, G_{2k-1}\right)$.
An n-chain is a sequence $G_1, G_2, G_3, \ldots, G_{2n}$ of finite play of the game.

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- **Winning Strategy & Category:** Let $G = \bigcap_{n\in\mathbb{N}} \bigcup \mathcal{H}_n$.
  - Player 2's winning strategy guarantees $G \subseteq C$.
  - Thus, $M \subseteq G^c$ is of effective first category in X.

Liouville numbers

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The set of Liouville numbers is "small" - has measure 0, and even (effective) Hausdorff dimension 0 [Staiger]. But its *complement* is effectively meager!

$$\bigcup_{n \in \mathbb{N}} \bigcap_{p \in \mathbb{Z}, q \in \mathbb{Z} \setminus \{0\}} B_{q^{-n}}\left(\frac{p}{q}\right).$$

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The set of Liouville numbers is topologically large, measure-theoretically small.

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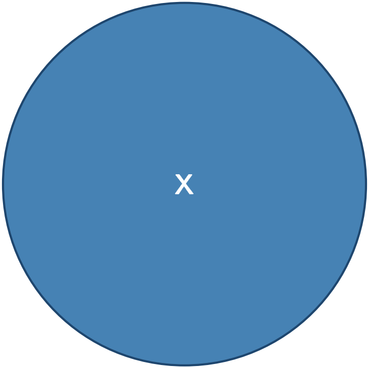
Wandering Set

An open set $E \subseteq X$ is said to be *wandering* if the sets in the sequence $\{T^{-i}E\}_{i\geq 0}$ are mutually disjoint.

Spaces with wandering open sets



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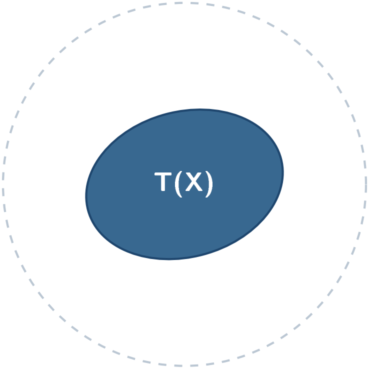

Poincaré Recurrence

Effective Poincaré Recurrence Theorem for Category

For a bounded c.e. open region $X \subseteq \mathbb{R}^n$, any computable homeomorphism T onto itself, with no non-empty wandering open set, all the points of X except a set of effective first category, are recurrent under T.

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Proof:

- Let $E \subsetneq X$ be a c.e. open set.
- Let $N_{\geq n}(E) = \{x \in E : (\forall j > n)(T^{-j}x \notin E)\}$ be points which do not recur after n.

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- Let $R_n = E \cap T^{-n}E$, the set of points in E which recur at the n^{th} level.

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  Then $R_n \cap N_{\geq n}(E) = \varnothing$.

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- Let $R_n = E \cap T^{-n}E$, the set of points in E which recur at the n^{th} level.

  Then $R_n \cap N_{\geq n}(E) = \varnothing$.
  $\implies N_{\geq n}(E)$ is not dense in E.

Proof (cont...):

- Consider the game $BM\langle N(E), X \setminus N(E)\rangle$.
- At stage k, if P_1 plays the set G_{2k-1}, P_2 plays G_{2k}, the first set in the enumeration of $G_{2k-1} \setminus \overline{N_{\geq k}(E)}$.

  (The choice of a non-empty set is guaranteed by the fact that $N_{\geq k}(E)$ is not dense in E.)
- Since $N(E) \cap \left(\bigcap_{k\in\mathbb{N}} G_k\right) = \varnothing$, P_2 wins the game.
- Hence $N(E)$ is of effective first category. $\square$

Effective Banach Category Theorem

For every set $A \subseteq X$ of a strongly computable T_0 space (X, τ, β, ν) that is of effective second category, there is a non-empty c.e. open set $G \subseteq X$ such that A is of effective second category at every point of G.

Thank You!