Constructive Dimension via Φ-Coverings and Faithfulness of Cantor Coverings

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Constructive Dimension

Definition (Lutz ('00), Mayordomo ('02))

For any $X \in \Sigma^\infty$, $\Sigma = \{0, 1\}$

$$\mathrm{cdim}(X) = \liminf_{n\to\infty} \frac{K(X \upharpoonright n)}{n}$$

where $K(\cdot)$ is *Kolmogorov Complexity*.

Constructive Dimension

Definition (Lutz and Mayordomo '08)

For an $x \in \mathbb{R}^n$, define

$$\mathrm{cdim}(x) = n \cdot \mathrm{cdim}(\mathrm{bin}(x)).$$

Eg $\mathrm{bin}(x, y) = \mathrm{bin}(x) \oplus \mathrm{bin}(y)$

$\mathrm{bin}(x) = b_0, b_1, \ldots$ s.t $x = \sum_i b_i \cdot 2^{-i}$.

Representation, Dimension and Faithfulness

What if we use a different representation Φ ?

- Can we formalize the notion of Φ-dimension ?
  - What properties do Φ need so that (analogues) of theorems from constructive dimension still hold ?
- Does Φ change constructive dimension ?
  - Faithfulness : $\forall x$, $\mathrm{cdim}_\Phi(x) = \mathrm{cdim}(x)$?

Overview

- Φ-dimension: Framework to study constructive dimension w.r.t representations.
- Cantor Series: Faithfulness depends on terms in series (log-limit condition).
- Point-to-set principle: Lift results to Hausdorff dimension faithfulness.

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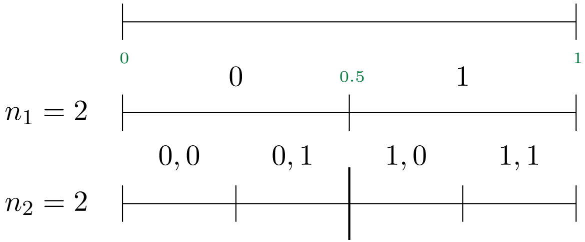
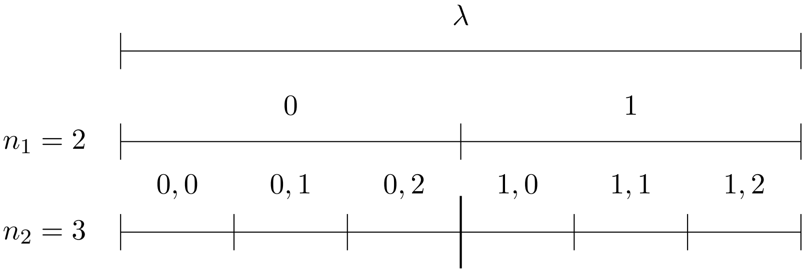
Base-2 representation for $r \in [0, 1]$



Cantor Series Representation

Given a sequence $Q : < n_i \in \mathbb{N} \setminus \{1\} >_{i \in \mathbb{N}}$.



Cantor Series Representation

Definition (Cantor 1869)

Given $Q = (n_k)_{n\geq 1}$ with $n_k \in \mathbb{N} \setminus \{1\}$ for all k, the Cantor series representation of $x \in [0, 1]$ is

$$x = \frac{a_1}{n_1} + \frac{a_2}{n_1 n_2} + \frac{a_3}{n_1 n_2 n_3} + \cdots ,$$

where each $a_k \in \{0, 1, \ldots, n_k - 1\}$.

Example: $n_k = (k + 1)!$

$e - 2 = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} \ldots$

Cantor Series Dimension

Theorem (Akhil, Nandakumar, Pulari [2])

For any $x \in [0,1]$, *and any computable Cantor coverings* Φ_Q *generated by* $Q = \{n_k\}_{k \in \mathbb{N}}$,

$$\mathrm{cdim}_{\Phi_Q}(x) = \liminf_{k \to \infty} \frac{K(X \restriction m_k)}{m_k},$$

where X *is a binary expansion of* x *and*

$$m_k = \lfloor \log_2(n_1 n_2 \cdots n_k) \rfloor .$$

cdim at Sparse Indices

$$\mathrm{cdim}(X) = \liminf_{n\to\infty} \frac{K(X{\upharpoonright}n)}{n}$$

$$\begin{array}{ccccccccccccc}
X = & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & \ldots \\
n & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
m_k & \downarrow & & \downarrow & & & \downarrow & & & \downarrow & &
\end{array}$$

$$\mathrm{cdim}_\Phi(X) = \liminf_{k\to\infty} \frac{K(X{\upharpoonright}m_k)}{m_k}$$

Faithfulness

Definition (Akhil, Nandakumar, Pulari [2])

We say that Φ_Q is faithful to constructive dimension iff

$$\forall x \in [0,1], \quad \mathrm{cdim}_{\Phi_Q}(x) = \mathrm{cdim}(x).$$

cdim at Sparse Indices

$$\mathrm{cdim}(X) = \liminf_{n\to\infty} \frac{K(X\restriction n)}{n} \qquad\qquad \mathrm{cdim}_{\Phi_Q}(X) = \liminf_{k\to\infty} \frac{K(X\restriction m_k)}{m_k}$$

$$\begin{array}{rcccccccccccc}
X = & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & \dots \\
n & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
m_k & \downarrow & & \downarrow & & & \downarrow & & & \downarrow & &
\end{array}$$

- How sparse can m_k be so that $\mathrm{cdim}_{\Phi_Q}(x) = \mathrm{cdim}(X)$?

Close indices cannot alter dimension

Landmark Lemma: $K(X \upharpoonright m_{k+1}) \leq K(X \upharpoonright m_k) + \Delta m_k + o(m_{k+1})$.

Trimming Inequality: $K(X \upharpoonright m_k) \leq K(X \upharpoonright m_{k+1}) + o(m_k)$.

Lemma

If $\lim_k \frac{m_{k+1}}{m_k} = 1$, *then for all* x, $\mathrm{cdim}_{\Phi_Q}(x) = \mathrm{cdim}(x)$.

Faraway indices can alter dimension

Lemma

If $\limsup \frac{m_{k+1}}{m_k} \neq 1$, *then there exists an* x *s.t* $\mathrm{cdim}_{\Phi_Q}(x) \neq \mathrm{cdim}(x)$.

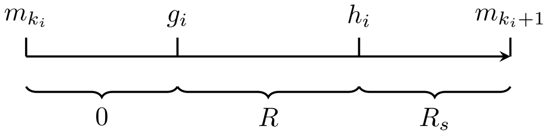


Figure: Filling of X during stage i.

Faithfulness

Theorem (Akhil, Nandakumar, Pulari '24)

A family of computable Cantor coverings Φ_Q generated by $Q = \{n_k\}_{k \in \mathbb{N}}$ is faithful with respect to constructive dimension if and only if:

$$\lim_{k \to \infty} \frac{\log n_k}{\log(n_1 n_2 \cdots n_{k-1})} = 0.$$

Corollary : All base-b representations are faithful!
[Hitchcock and Mayordomo '13]

Φ-Covers

Definition

Let $(\mathbb{X}, d)$ be a metric space. A countable family $\Phi \subseteq 2^{\mathbb{X}}$ is called a family of covering sets over $\mathbb{X}$ if it satisfies:

- **Fineness:** For every $x \in \mathbb{X}$ and every $\delta > 0$, there exists $U \in \Phi$ such that

$$x \in U \qquad \text{and} \qquad |U| < \delta.$$

Constructive Φ-dimension : Covers

Fix an enumeration $\delta : \mathbb{N} \to \Phi$.

Definition

A set $\mathcal{F} \subseteq \mathbb{X}$ has an effective s-dimensional Φ-null cover if there exists a Turing machine $M : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for every $r \in \mathbb{N}$:

$$\sum_i |\delta(M(i,r))|^s \leq 2^{-r} \quad \text{and} \quad \bigcup_i \delta(M(i,r)) \supseteq \mathcal{F}.$$

Definition

For $\mathcal{F} \subseteq \mathbb{X}$,

$$\mathrm{cdim}_\Phi(\mathcal{F}) = \inf \left\{ s \geq 0 : \mathcal{F} \text{ has an effective } s\text{-dimensional } \Phi\text{-null cover} \right\}.$$

Kolmogorov Complexity

Definition

Given an $r \in \mathbb{N}$ and $x \in \mathbb{X}$, define

$$K_r(x, \Phi) = \min_{U \in \Phi} \{K(\delta^{-1}(U)) : x \in U \text{ and } |U| \leq 2^{-r}\}.$$

If for all $w \in \mathbb{N}$, $|\delta(w)|$ is computable. Then,

Theorem

For any $\mathcal{F} \subseteq \mathbb{X}$,

$$\operatorname{cdim}_\Phi(\mathcal{F}) = \sup_{x \in \mathcal{F}} \operatorname{cdim}_\Phi(x).$$

For any $x \in \mathbb{X}$,

$$\operatorname{cdim}_\Phi(x) = \liminf_{r \to \infty} \frac{K_r(x, \Phi)}{r}.$$

Hausdorff Dimension

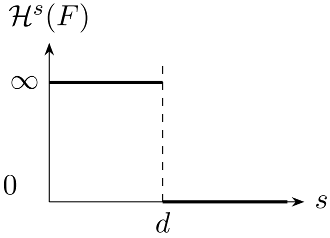
Given Euclidean space $\mathbb{R}^n$ and a set $\mathcal{F} \subseteq \mathbb{X}$,

- **δ-cover of $\mathcal{F}$:** $\{U_i\}_{i\in\mathbb{N}} \supseteq \mathcal{F}$ with $U_i \subseteq \mathbb{X}$ and $|U_i| \leq \delta$ for all i.

- **s-dim. Hausdorff (outer) measure:**

$$\mathcal{H}^s_\delta(\mathcal{F}) = \inf\left\{\sum_i |U_i|^s : \{U_i\}_{i\in\mathbb{N}} \text{ is a } \delta\text{-cover of } \mathcal{F}\right\},$$

$$\mathcal{H}^s(\mathcal{F}) = \lim_{\delta\to 0^+} \mathcal{H}^s_\delta(\mathcal{F}).$$



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$$\mathcal{H}^s(\mathcal{F}) = \lim_{\delta\to 0^+} \mathcal{H}^s_\delta(\mathcal{F}).$$

- **Hausdorff dimension:**

$$\dim(\mathcal{F}) = \inf\{s \geq 0 : \mathcal{H}^s(\mathcal{F}) = 0\}.$$

Hausdorff Φ-Dimension

Given a metric space $\mathbb{X}$, a family of covering sets Φ over $\mathbb{X}$, and a set $\mathcal{F} \subseteq \mathbb{X}$,

- **δ-cover of $\mathcal{F}$ using Φ:** $\{U_i\}_{i\in\mathbb{N}} \supseteq \mathcal{F}$ with $U_i \in \Phi$ and $|U_i| \leq \delta$ for all i.
- **s-dim. Hausdorff (outer) measure:**

$$\mathcal{H}^s_\delta(\mathcal{F}, \Phi) = \inf\left\{\sum_i |U_i|^s : \{U_i\}_{i\in\mathbb{N}} \text{ is a } \delta\text{-cover of } \mathcal{F} \text{ using } \Phi\right\},$$

$$\mathcal{H}^s(\mathcal{F}, \Phi) = \lim_{\delta\to 0^+} \mathcal{H}^s_\delta(\mathcal{F}, \Phi).$$

- **Hausdorff Φ-dimension:**

$$\dim_\Phi(\mathcal{F}) = \inf\{s \geq 0 : \mathcal{H}^s(\mathcal{F}, \Phi) = 0\}.$$

Point-to-set principle for Φ-dimension

Theorem

Let Φ be a family of covering sets over $\mathbb{X}$. For all $\mathcal{F} \subseteq \mathbb{X}$,

$$\dim_{\Phi}(\mathcal{F}) = \min_{A \subseteq \mathbb{N}} \sup_{x \in \mathcal{F}} \liminf_{r \to \infty} \frac{K_r^A(x, \Phi)}{r}.$$

Note : Computability of diameters of Φ is not required !

Hausdorff Faithfulness

Definition

Φ is faithful to hausdorff dimension iff

$$\forall \mathcal{F} \subseteq \mathbb{X}, \quad \dim_{\Phi}(\mathcal{F}) = \dim(\mathcal{F}).$$

Hausdorff faithfulness of Cantor Coverings

Lemma (Albeverio, Ivanenko, Lebid and Torbin '20 [1])

A family of Cantor coverings Φ_Q *generated by* $Q = \{n_k\}_{k \in \mathbb{N}}$ *is faithful with respect to Hausdorff dimension if and only if*

$$\lim_{k \to \infty} \frac{\log n_k}{\log n_1 \cdot n_2 \ldots n_{k-1}} = 0.$$

We reprove using Kolmogorov Complexity !

Conclusion

- Formalize Φ-dimension framework.
- Study (constructive) faithfulness of Cantor Coverings.
- Point-to-set principle: Carry faithfulness results to Hausdorff dimension.

Open Questions

- For a class of coverings Φ (with computable diameters)

  Φ is faithful to constructive dimension $\iff$ Φ is faithful to Hausdorff dimension ?

- (Constructive) faithfulness of other coverings Φ.
- Strong dimension analogues of Φ-dimension and faithfulness ?

References

S. Albeverio, Ganna Ivanenko, Mykola Lebid, and Grygoriy Torbin.
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