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Complexity Bounds for Illumination in 2D using Reflections

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Presented by:
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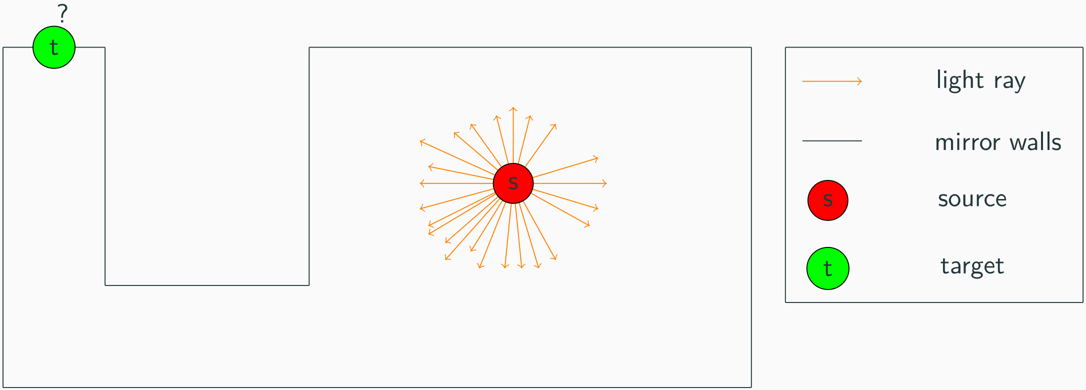
Agenda

Introduction

We look at a naturally interesting problem, which is the Illumination problem. Given a light source (not a fixed ray), we inquire whether a specific point is illuminated. In other words, does some ray from the source reach the target?

- other people - looked at this problem before
- we look at the complexity aspect in specific settings

Illustration of the generic Illumination problem



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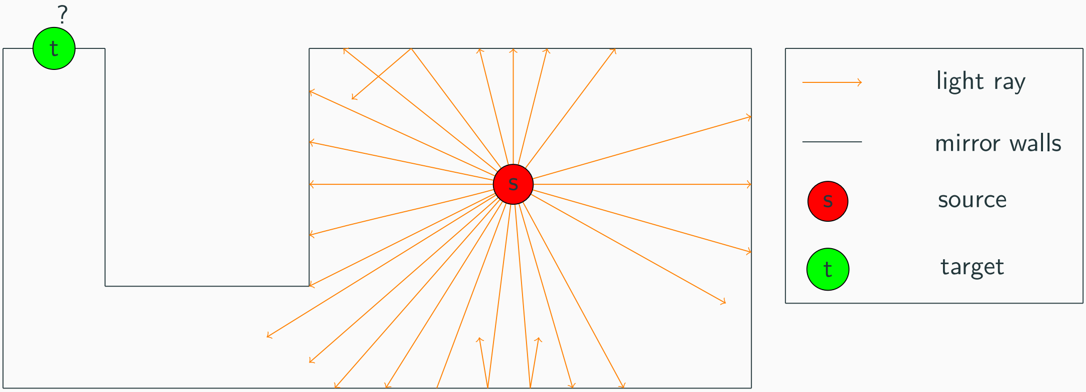

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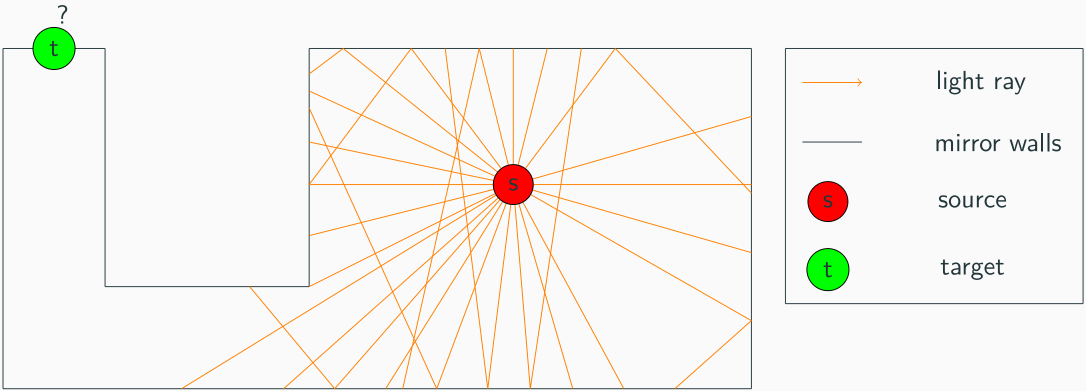

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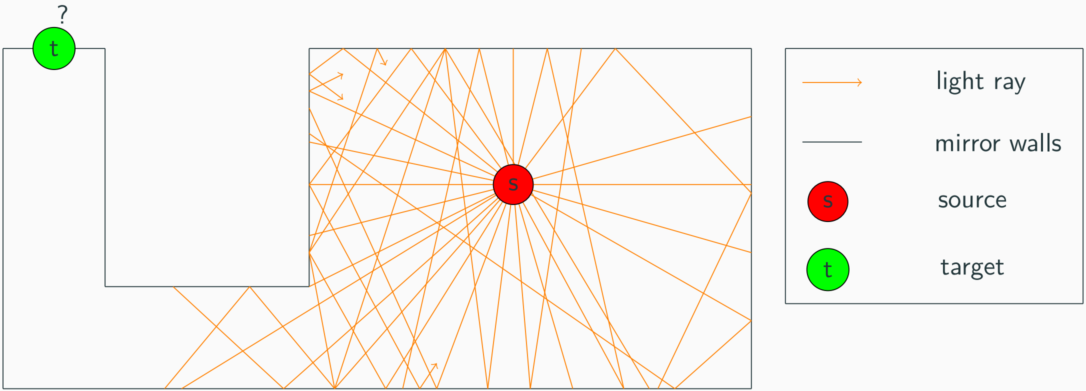

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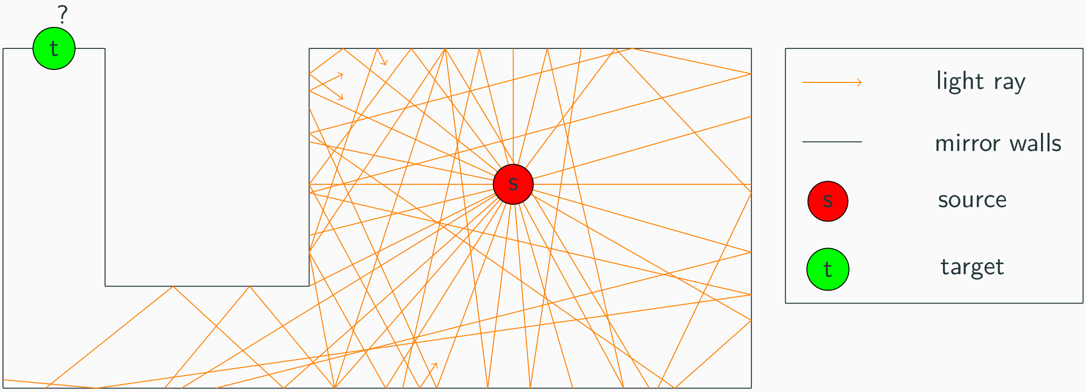

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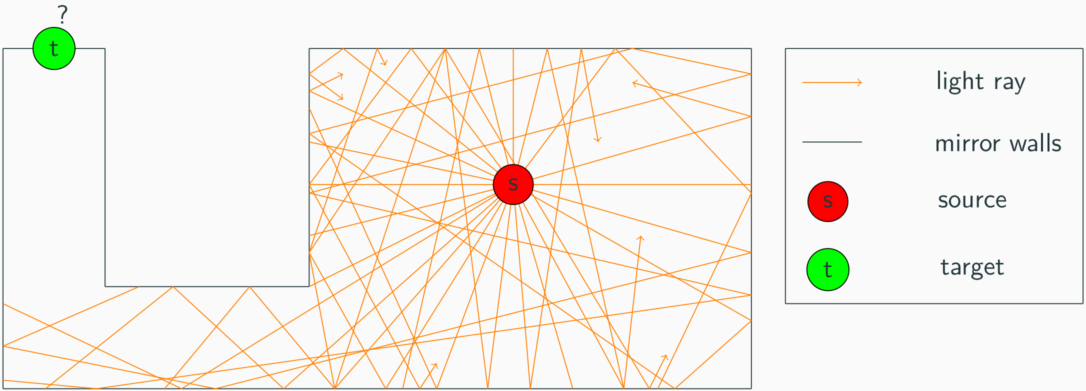

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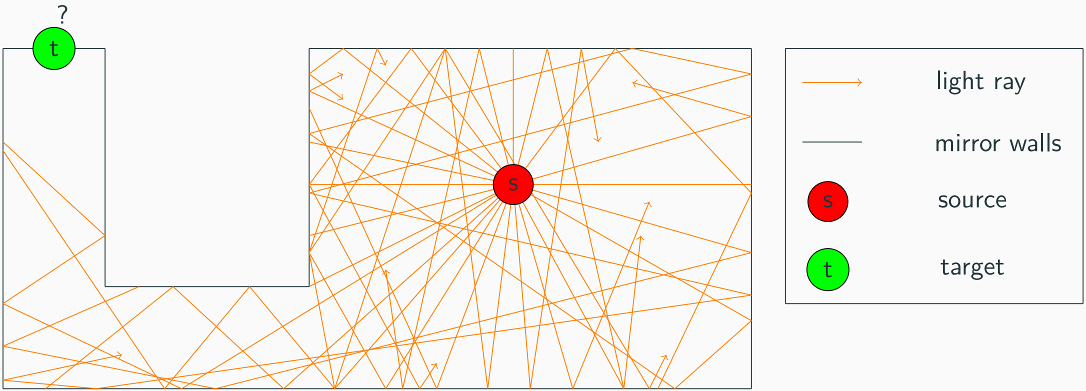

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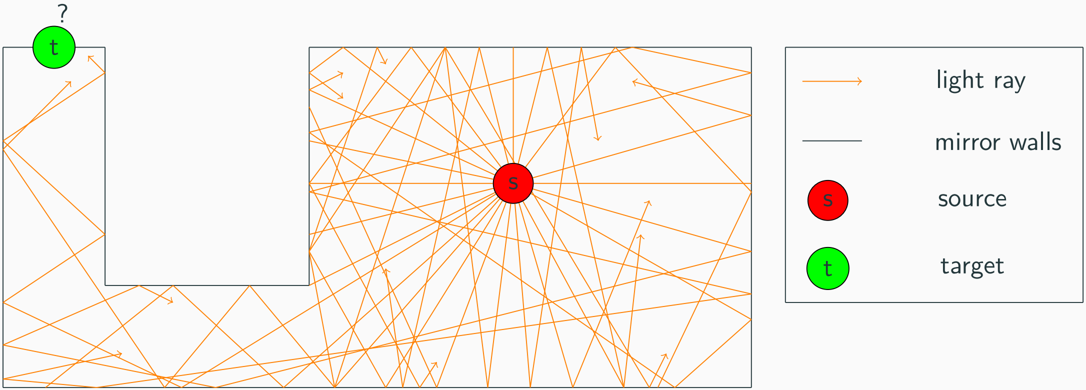

Related Works

- Penrose et.al [**Penrose1958**]
  - illumination problem - enclosed regions

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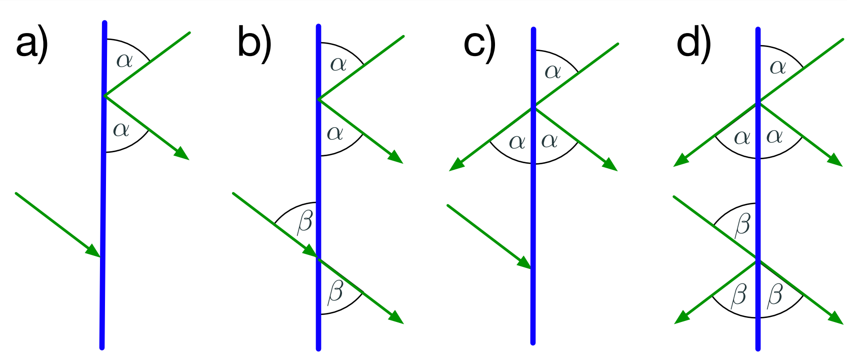
- Penrose et.al [**Penrose1958**]
  - illumination problem - enclosed regions
  - curved (elliptic and straight) surface - light source cannot illuminate all points within
- Aronov et.al [**Aronov1998**]
  - an exponential time bounded algorithm
  - given an n-sided polygon with mirrored walls, light source
  - compute the entire region illuminated after k reflections and all the corresponding reflection paths
  - Illumination problem - exponential complexity for a linear number of reflections

Components



Full mirrors, one-way merging mirrors, one-way splitting mirrors as well as one-way mirrors

Problem Definition

Definition (The Illumination Problem)

We consider the Illumination problem as the question, whether a ray can be emitted from a source point, that reaches a given target point, while passing or being reflected by a given set of mirrors.

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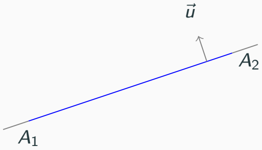
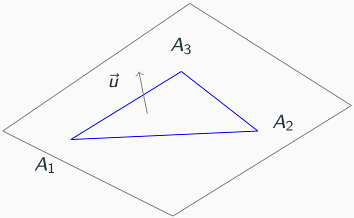
We consider the Illumination problem as the question, whether a ray can be emitted from a source point, that reaches a given target point, while passing or being reflected by a given set of mirrors. We may limit the number t of reflections the rays needs to reach the target for this problem and call this variant t-bounded Illumination.

Mirrors and Reflections I



mirror = 1-simplex, A_1A_2



mirror = 2-simplex, the triangle, $A_1A_2A_3$

- Each mirror is a k-simplex, $L_i = (A_{i,1}, \ldots, A_{i,k+1})$ with rational coordinates, lying in a hyperplane of (k+1)-space.
- Given a k-dimensional simplex in k + 1-dimensional space we can determine an orthogonal vector $\vec{u}$, for the k-dimensional hyperplane given by the vectors $\vec{v}_i$ of the simplex, by solving the k equations for $i \in \ 1, ..., k$
- All vectors, $\vec{v}_i = \overline{A_1 A_{i+1}}$ from A_1 to A_{i+i} for $i \in \ 1, ..., k$ are linearly independent

Mirrors and Reflections II

- coordinates of the points $A_1, ..., A_{k+1}$ are given by quotients of binary or integer numbers (for both the numerator and denominator)
- we determine the coordinates of $\vec{u}$ in polynomial time
- the representation of these coordinates require at most polynomial space.
- We have, $\vec{v}_i \cdot u = 0$ where $\vec{v}_i \cdot u$ denote the inner product of $\vec{u}$ and $\vec{v}_i$.
- In the case of k-dimensional simple, merging or splitting mirrors, we assume that $\vec{u}$ also determines the direction in which the reflection occurs.

Reflection formulae I

- If we assume that the k-dimension hyperplane $L_i = (A_{i,1}, \ldots, A_{i,k+1})$ with an orthogonal vector $\vec{u}_i$ includes the origin, then the simple reflection S' of a point S can be determined by

$$S' \;=\; S - 2\,\frac{\vec{u}_i \cdot S}{\vec{u}_i \cdot \vec{u}_i}\,\vec{u}_i.$$

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- If the origin is not a point of the hyperplane then we have to shift the coordinate system before and after the reflection by a point of the simplex, e.g.

$$S' \;=\; \left[(S - A_{i,1}) - 2 \cdot \frac{\vec{u}_i \cdot (S - A_{i,1})}{\vec{u}_i \cdot \vec{u}_i} \cdot \vec{u}_i\right] + A_{i,1} \;=\; S - 2 \cdot \frac{\vec{u}_i \cdot S}{\vec{u}_i \cdot \vec{u}_i} \cdot \vec{u}_i + 2 \cdot \frac{\vec{u}_i \cdot A_{i,1}}{\vec{u}_i \cdot \vec{u}_i} \cdot \vec{u}_i.$$

Reflection formulae II

- Let

$$u_i = \frac{2}{\vec{u}_i \cdot \vec{u}_i} \cdot \vec{u}_i \qquad \text{and} \qquad \hat{u}_i = \frac{2 \cdot \vec{u}_i \cdot A_{i,1}}{\vec{u}_i \cdot \vec{u}_i} \cdot \vec{u}_i$$

  denote two scaled versions of the vector $\vec{u}_i$, and for simplifications we will assume that both vectors are also part of the simplex description, then

$$S' \ = \ S - (u_i \cdot S) \cdot \vec{u}_i + \hat{u}_i. \tag{1}$$

- To determine whether the reflection of the point S is possible, i.e. it is located at the correct side of the hyperplane defined by the mirror, we have only to verify whether

$$\overrightarrow{S'S} \cdot u_i \ > \ 0. \tag{2}$$

- To get the new direction of the ray after the reflection we determine an additional point on the ray, e.g. $H = S + \vec{r}$, then we determine the reflected position H' of H by using Equation (1). Finally the new direction $\vec{r}\,'$ of the ray after the reflection will be

$$\vec{r}\,' \ = \ \overrightarrow{S'H'}. \tag{3}$$

Intersection point

- We could determine also the intersection point P of the ray with the mirror. For this we have to find the values $c_1, \ldots, c_k, c_{k+1}$ such that

$$\sum_{j=1}^{k} c_j \cdot \overrightarrow{A_{i,1}A_{i,j}} - c_{k+1} \cdot \vec{r} \;=\; 0. \tag{4}$$

Then we get the intersection point

$$P \;=\; c_{k+1} \cdot \vec{r}. \tag{5}$$

To test whether the intersection point is within the mirror, we have to verify that

$$\forall j \in \{1, \ldots, k\} : \; c_j \in [0..1]. \tag{6}$$

Iterating over virtual sources

- For every reflection the number of bits of the numerators and denominators increase by a value at most linear in the input size.
- Hence, if the number of reflections - polynomial in the input size - the values remain polynomial in the input size.
- Replace the relative source positions by the intersection points
- might double the length of the binary representation of the numerator, denominator with each reflection
- resulting in exponential large binary representations for a polynomial number of reflections.
- we should not iterate over the intersection points.

Ray Tracing as well as Non-determinism in the Illumination problem

- P : simple/full mirrors and merging mirrors, a polynomial number of reflections, Ray Tracing - solvable in polynomial time.
- Sequence of mirrors the ray hits is deterministic.
- NP : add the splitting/one-way mirrors - a beam may or may not pass through a given mirror
- Nondeterministic branch point.

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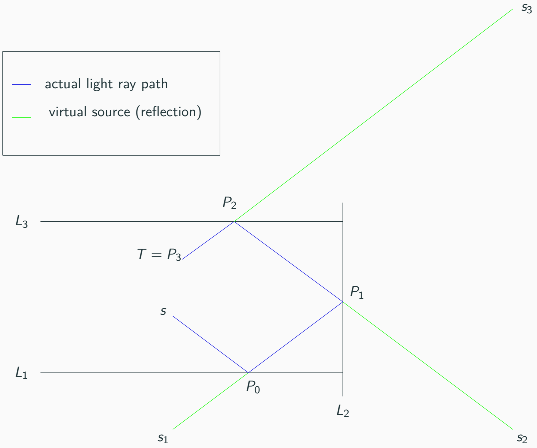
- Sequence of mirrors the ray hits is deterministic.
- NP : add the splitting/one-way mirrors - a beam may or may not pass through a given mirror
- Nondeterministic branch point.
- For Illumination - source point, s and target point, t
- one of infinitely many light rays emitted from source - all directions
- guess combinatorial part - sequence of mirrors in bounded reflection

Illumination Upper bound

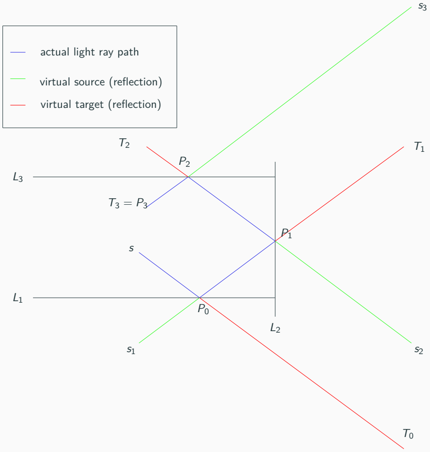


Notation: P_{i-1} denotes the hit point on mirror L_i, and $P_\ell = T$. We write $\mathcal{R}_i(X) = X - (u_i \cdot X)\,\vec{u}_i + \hat{u}_i$ for the reflection of X at L_i, and $L_i \cap \overline{XY}$ for the point at which the line through X and Y meets the simplex L_i

Bounded illumination I

1. Guess a sequence $L_1, \ldots, L_\ell$ of mirrors with $\ell \le t$, and set $S_0 := S$ and $P_\ell := T$.
2. For $i = 1, \ldots, \ell$ let $S_i := \mathcal{R}_i(S_{i-1})$ be the image of the source after the first i reflections.
3. For $i = \ell, \ell-1, \ldots, 1$ let $P_{i-1} := L_i \cap \overline{S_i P_i}$, and reject if this point is undefined or if some simplex L_j blocks the direct connection from P_{i-1} to P_i.
4. Accept if no simplex blocks the direct connection from S to P_0, and reject otherwise.

Illumination Upper bound



Bounded illumination II

1. Guess a sequence $L_1, \ldots, L_\ell$ of mirrors with $\ell \leq t$, and set $S_0 := S$, $T_\ell := T$ and $P_\ell := T$.
2. For $i = 1, \ldots, \ell$ reject if $\vec{u}_i \cdot (S_{i-1} - A_{i,1}) \leq 0$, i.e. if S_{i-1} lies on the non-reflective side of L_i; otherwise let $S_i := \mathcal{R}_i(S_{i-1})$.
3. For $i = \ell - 1, \ell - 2, \ldots, 0$ reject if T_{i+1} lies on the non-reflective side of L_{i+1}; otherwise let $T_i := \mathcal{R}_{i+1}(T_{i+1})$.
4. For every $i \in \{1, \ldots, \ell\}$, independently and in any order, let $P_{i-1} := L_i \cap \overline{S_i T_i}$, and reject if this point is undefined.
5. Reject if for some i a simplex L_j blocks the direct connection from P_{i-1} to P_i.
6. Accept if no simplex blocks the direct connection from S to P_0, and reject otherwise.

Illumination Upper Bound - Summary

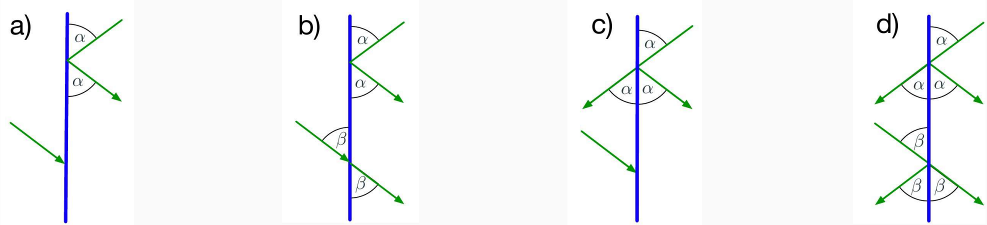
- If bouncing points depend on each other -
  - the numerator and denominator bit lengths - double in each step
  - $O(2^\ell)$-size numbers after ℓ reflections.

Illumination Upper Bound - Summary

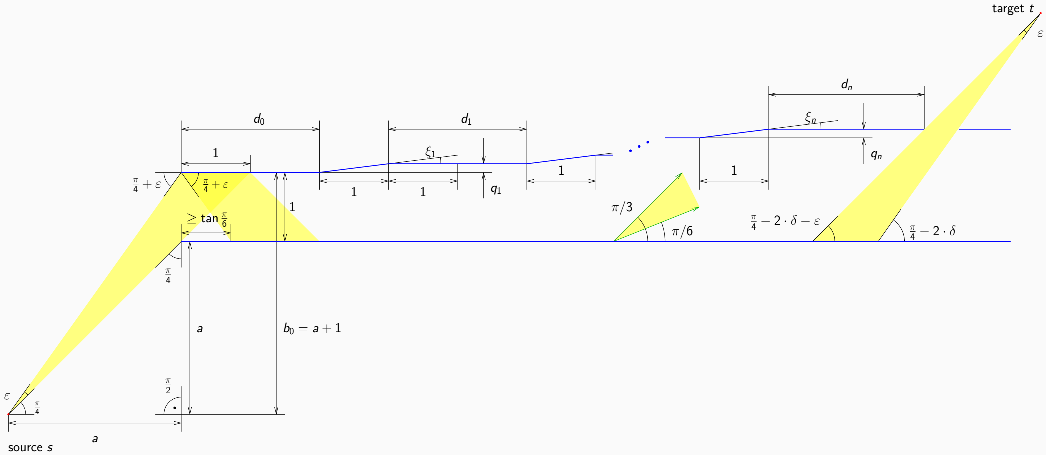
- If bouncing points depend on each other -
  - the numerator and denominator bit lengths - double in each step
  - $O(2^{\ell})$-size numbers after ℓ reflections.
- Only chained quantities are S_i and T_i
  - rational mirror data independent of the point being reflected
  - results in purely additive bit growth.

Theorem I

Polynomially bounded Illumination is in NP for polynomial dimensions in the context of full plane, one-way merging, one-way splitting as well as one-way mirrors for polynomial number of reflections.



Illumination Lower Bound - Case I - Exponential Reflections



The encoding process I

We start with a Subset Sum instance: weights $x_1, \ldots, x_n$ and a target integer B. We ask whether some subset $X' \subseteq \{1, \ldots, n\}$ satisfies

$$\sum_{i \in X'} x_i = B.$$

The reduction turns each weight x_i into a small angle

$$\xi_i \; = \; \frac{\pi}{24} \cdot \frac{x_i}{\sum_i x_i},$$

and turns the target B into a corresponding angle

$$\delta \; = \; \frac{\pi}{24} \cdot \frac{B}{\sum_i x_i}.$$

The encoding process II

We have this common constant $\frac{\pi}{24\sum_i x_i}$, and therefore an equivalence:

$$\sum_{i \in X'} x_i = B \quad \Longleftrightarrow \quad \sum_{i \in X'} \xi_i = \delta.$$

So the subset sums to B and the corresponding angles sum to δ are both equivalent

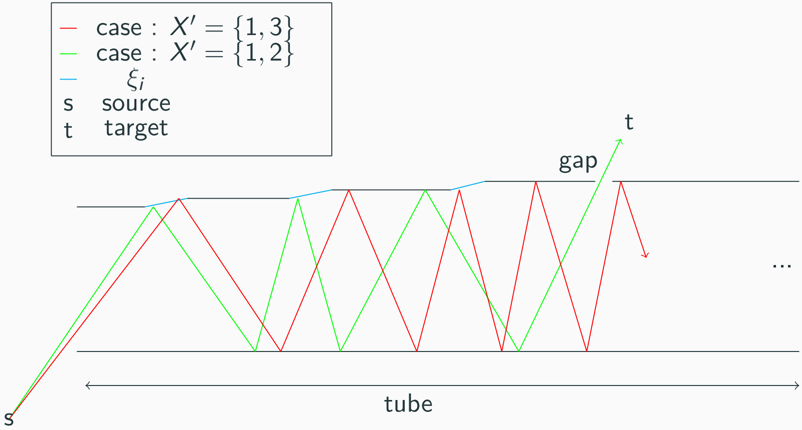
Example encoding I

$X = \{1, 2, 4\}, \quad B = 3$. Subset Sum question: given weights $x_1 = 1$, $x_2 = 2$, $x_3 = 4$ and target $B = 3$, is there a subset $X' \subseteq \{1, 2, 3\}$ (which are the indices of the elements of X) whose weights sum to exactly B? Enumerate all $2^3 = 8$ subsets -

X'	sum	$B = 3$?
$\emptyset$	0	no
$\{1\}$	1	no
$\{2\}$	2	no
$\{3\}$	4	no
$\{1, 2\}$	3	**yes**
$\{1, 3\}$	5	no
$\{2, 3\}$	6	no
$\{1, 2, 3\}$	7	no

target if and only if such a subset exists - meaning that the target will be illuminated, and the beam that reaches the target is precisely the one corresponding to picking elements 1 and 2.

Example encoding II



The angular encoding I

The elements of X can't be used as is, but need to be turned into angles. The paper defines

$$\xi_i = \frac{\pi}{24} \cdot \frac{x_i}{\sum_i x_i}, \qquad \delta = \frac{\pi}{24} \cdot \frac{B}{\sum_i x_i}.$$

With $\sum_i x_i = 1 + 2 + 4 = 7$, and measuring everything in units of $\frac{\pi}{168}$ (since $\frac{\pi}{24} \cdot \frac{1}{7} = \frac{\pi}{168}$), we get:

$$\xi_1 = 1, \quad \xi_2 = 2, \quad \xi_3 = 4, \quad \delta = 3 \qquad \text{(all in units of } \tfrac{\pi}{168}\text{)}.$$

establishing equivalence :

$$\sum_{i \in X'} x_i = B \quad \Longleftrightarrow \quad \sum_{i \in X'} \xi_i = \delta.$$

The angular encoding II

- $\xi_1 + \xi_2 = 1 + 2 = 3 = \delta$.
  - which subset of angles sums to δ?
  - exactly the same answer, i.e, $\{1, 2\}$

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- $\xi_1 + \xi_2 = 1 + 2 = 3 = \delta$.
  - which subset of angles sums to δ?
  - exactly the same answer, i.e, $\{1, 2\}$
- $\frac{\pi}{24}$ small enough
- all angles satisfy the geometric bounds in the proof (i.e, the $\pi/4$ to $\pi/3$ window) holds.
- guarantees $\sum_i \xi_i \leq \frac{\pi}{24}$
- keeps beam's cumulative deflection small and controlled.

Exponentially many reflections

- The cone must be **wide enough** at every stage
- Each cone to expand its horizontal width from 1 up to (at least) 5 units before reaching the next tilted mirror.
- 5 comes from the beam-spread bound

$$2h_{\max}\cot(\pi/6) = 2\sqrt{3}\,h_{\max} \leq \frac{8\sqrt{3}}{3} \leq 5, \qquad \text{using } h_{\max} \leq \frac{4}{3}.$$

- To make the cone wider - how far back the virtual source appears to be at stage i - via Thales's theorem on the widening triangle of the cone -

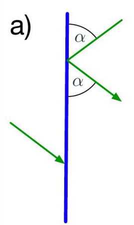
$$b_i = 5\,b_{i-1} + 5c$$

  where c is a small constant.
- Each stage multiplies the previous distance by 5. Starting from b_0, after n stages one obtains

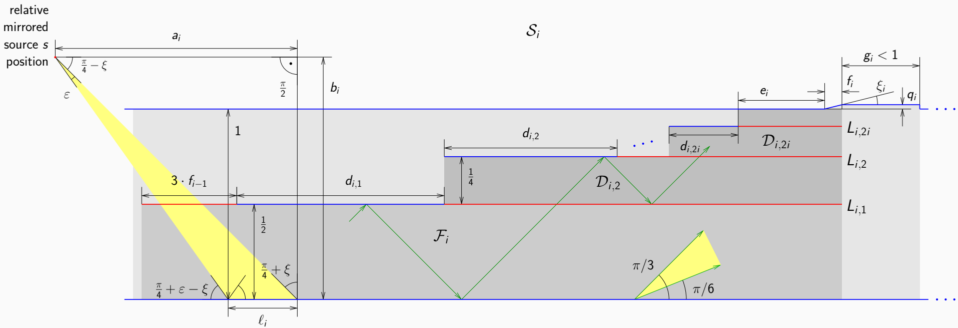
$$b_n \approx b_0 \cdot 5^{\,n} \in b_0 \cdot 2^{O(n)}.$$

Theorem II

Illumination in 2D with simple plane mirrors is NP-hard,if we allow exponentially many reflections.



Illumination lower bound - Case II - Linear reflections



The local shrinking strategy

Concretely, for the sub-construction $\mathcal{S}_i$ handling tilted mirror ξ_i:

- The tilted mirror's horizontal footprint is shrunk from 1 down to

$$f_i = 2^{-2i},$$

  so its height contribution becomes

$$q_i = \frac{1}{2^{2i}} \cdot \tan(\xi_i)$$

  instead of $q_i = \tan(\xi_i)$
- The active sub-tube height near ξ_i is likewise shrunk to 2^{-2i}.

The shrinking footprints

In this new gadget, the tilted mirror for element i has horizontal footprint

$$f_i = 2^{-2i}.$$

For our three elements from the example:

$$f_1 = 2^{-2} = \frac{1}{4} = 0.25, \qquad f_2 = 2^{-4} = \frac{1}{16} = 0.0625, \qquad f_3 = 2^{-6} = \frac{1}{64} \approx 0.0156.$$

- the local sub-tube height near mirror ξ_i is shrunk to the same scale 2^{-2i}.

The shrinking footprints

In this new gadget, the tilted mirror for element i has horizontal footprint

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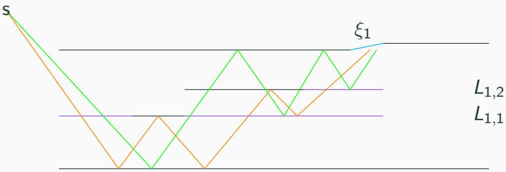
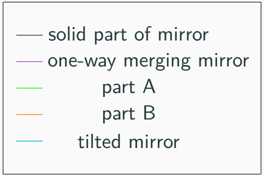
- the local sub-tube height near mirror ξ_i is shrunk to the same scale 2^{-2i}.
- the stage-boxes $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ progressively *shorter* - the whole apparatus telescoping downward in scale as one moves right
- shrinking the footprint - shrinks the mirror's height contribution: instead of $q_i = \tan(\xi_i)$ - we get $q_i = \frac{1}{2^{2i}} \tan(\xi_i)$

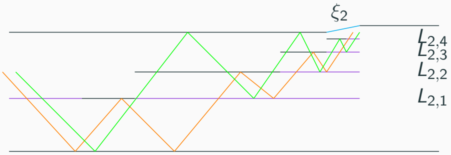
Example encoding

S_1 : 2i = 2 lines, sub-tube : 1 , $\frac{1}{2}$, $\frac{1}{4}$, tilted mirror width, $f_1 = 2^{-2}$





S_2 : 2i = 4 lines, sub-tube : 1 , $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$,tilted mirror width, $f_2 = 2^{-4}$



Why each line costs only a constant number of reflections ?

- lengths of these mirror segments chosen relative to the local shrunk scale
- set $d_{i,j} = 2^{6-j}$
- beam entering sub-tube $D_{i,j}$ - at least a constant number c of double-reflections before exiting, regardless of i
- the number of reflections within any single line $L_{i,j}$ - bounded by a constant
- compute this bound explicitly as

$$\frac{d_{i,j}}{\frac{1}{\sqrt{3}} \cdot 2^{-j}} + 12 \leq 2^6\sqrt{3} + 12 < 123.$$

Counting the total reflections

- stage $\mathcal{S}_i$ has $2i$ lines, each costing ≤ 123 reflections, plus a couple of extra for the top-most tube
- we bound $\mathcal{S}_i$ at $123 \cdot (2i + 2)$ reflections.
- summing over all stages $i = 1, \ldots, n$:

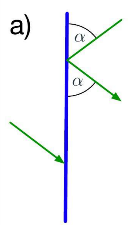
$$\sum_{i=1}^{n} 123\,(2i + 2) \;=\; 123\left(2 \cdot \frac{n(n+1)}{2} + 2n\right) \;=\; 123\,(n^2 + 3n) \;=\; \Theta(n^2).$$

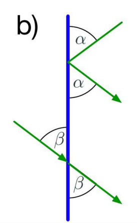
Why $\Theta(n^2)$ counts as NP

- use $\Theta(n^2)$ *mirrors* (each stage $\mathcal{S}_i$ needs $\Theta(i)$ mirror lines, summing to $\Theta(n^2)$),it uses $\Theta(n^2)$ *reflections*.
- number of reflections is **polynomial in the size of the mirror system**
- $\Theta(n^2)$ reflections is certainly *polynomial* in the input
- property that matters for landing in the NP membership

Theorem III

The Linear bounded Illumination problem using full plane mirrors and one-way merging mirrors is NP-hard (NP-Complete) for two dimensions and linear number of reflections.





Conclusion

- Summary of Results :

Illumination complexity	Dimensions	Type of mirrors	Number of reflections
$\mathcal{NP}$	1	a, b, c, d	polynomial
$\mathcal{NP}$-Hard	2	a	exponential
$\mathcal{NP}$-Hard	2	a + b	linear

Future Work

- considered the Illumination in 2D, for non-enclosed spaces
- explore if the lower/upper bound results hold with linear or exponential number of reflections - for enclosed spaces
- In [**fsttcs**], introduce and investigate the computational complexity - Pinball Wizard problem
- one-way gates (outswing doors), plane walls, parabolic walls, moving plane walls, and bumpers that cause acceleration or deceleration.
- the initial position and velocity of the pinball, the task is to decide - it will hit a specified target point.
- whether for other complex gadgets ?
- also using the ABCD model from [**ABCDlensmirror**]

References

- L Penrose and R Penrose. Puzzles for christmas. New Scientist, 25:1580–1581, 1958.
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Thank You !

Pre-requisites

- k-dimensional simplex is a generalisation of a triangle from 2 dimensions to k dimensions
- Each mirror is a k-simplex, $L_i = (A_{i,1}, \ldots, A_{i,k+1})$ with rational coordinates, lying in a hyperplane of (k+1)-space.
- Given a k-dimensional simplex in k + 1-dimensional space we can determine an orthogonal vector $\vec{u}$, for the k-dimensional hyperplane given by the vectors $\vec{v}_i$ of the simplex, by solving the k equations for $i \in \ 1, \ldots, k$
- All vectors, $\vec{v}_i = \overline{A_1 A_{i+1}}$ from A_1 to A_{i+i} for $i \in \ 1, \ldots, k$ are linearly independent
- in our setting, the coordinates of the points $A_1, \ldots, A_{k+1}$ are given by quotients of binary or integer numbers (for both the numerator and denominator), then we can determine the coordinates of $\vec{u}$ in polynomial time, and the representation of these coordinates require at most polynomial space.
- We have, $\vec{v}_i \cdot u = 0$ where $\vec{v}_i \cdot u$ denote the inner product of $\vec{u}$ and $\vec{v}_i$.
- In the case of k-dimensional simple, merging or splitting mirrors, we assume that $\vec{u}$ also determines the direction in which the reflection occurs.
- Now we define a reflection of a point S. For a k + 1-dimension we define the reflection on mirror given by a k-dimensional simplex by using the simple reflection on the corresponding k-dimension hyperplane.

Reflection formulae

- If we assume that the k-dimension hyperplane $L_i = (A_{i,1}, \ldots, A_{i,k+1})$ with an orthogonal vector $\vec{u}_i$ includes the origin, then the simple reflection S' of a point S can be determined by

$$S' \;=\; S - 2\,\frac{\vec{u}_i \cdot S}{\vec{u}_i \cdot \vec{u}_i}\,\vec{u}_i.$$

- If the origin is not a point of the hyperplane then we have to shift the coordinate system before and after the reflection by a point of the simplex, e.g.

$$S' \;=\; \left[(S - A_{i,1}) - 2 \cdot \frac{\vec{u}_i \cdot (S - A_{i,1})}{\vec{u}_i \cdot \vec{u}_i} \cdot \vec{u}_i\right] + A_{i,1} \;=\; S - 2 \cdot \frac{\vec{u}_i \cdot S}{\vec{u}_i \cdot \vec{u}_i} \cdot \vec{u}_i + 2 \cdot \frac{\vec{u}_i \cdot A_{i,1}}{\vec{u}_i \cdot \vec{u}_i} \cdot \vec{u}_i.$$

- Let

$$u_i = \frac{2}{\vec{u}_i \cdot \vec{u}_i} \cdot \vec{u}_i \qquad \text{and} \qquad \hat{u}_i = \frac{2 \cdot \vec{u}_i \cdot A_{i,1}}{\vec{u}_i \cdot \vec{u}_i} \cdot \vec{u}_i$$

  denote two scaled versions of the vector $\vec{u}_i$, and for simplifications we will assume that both vectors are also part of the simplex description, then

$$S' \; = \; S - (u_i \cdot S) \cdot \vec{u}_i + \hat{u}_i. \tag{7}$$

- Recall that we assume that for k-dimensional simple, merging or splitting mirrors, $\vec{u}_i$ also determines the direction in which the reflection of the mirror occurs. To determine whether the reflection of the point S is possible, i.e. it is located at the correct side of the hyperplane defined by the mirror, we have only to verify whether

$$\overrightarrow{S'S} \cdot u_i \; > \; 0. \tag{8}$$

- To get the new direction of the ray after the reflection, we can proceed as follows. First we determine an additional point on the ray, e.g. $H = S + \vec{r}$, then we determine the reflected position H' of H. Finally the new direction $\vec{r}\,'$ of the ray after the reflection will be

$$\vec{r}\,' \; = \; \overrightarrow{S'H'}. \tag{9}$$

- In some applications it might be useful to determine also the intersection point P of the ray with the mirror. For this we have to find the values $c_1, \ldots, c_k, c_{k+1}$ such that

$$\sum_{j=1}^{k} c_j \cdot \overrightarrow{A_{i,1}A_{i,j}} - c_{k+1} \cdot \vec{r} \; = \; 0. \tag{10}$$

Then we get the intersection point

$$P \; = \; c_{k+1} \cdot \vec{r}. \tag{11}$$

To test whether the intersection point is within the mirror, we have to verify that

$$\forall j \in \{1, \ldots, k\} : \; c_j \in [0..1]. \tag{12}$$

- For every reflection the number of bits of the numerators and denominators increase by a value at most linear in the input size.
- Hence, if the number of reflections - polynomial in the input size - the values remain polynomial in the input size.
- Replace the relative source positions by the intersection points - might double the length of the binary representation of the numerator an the denominator with each reflection, resulting in exponential large binary representations for a polynomial number of reflections.
- Therefore, we should not iterate over the intersection points.