

Completeness in computable topology

Arno Pauly

Swansea University

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Outline

Quasi-Polish spaces

Overt choice

Completions and preservation of properties

Multiretraceability

The initial question

Should we work with arbitrary separable metric spaces, or with complete ones?

- ▶ If the space is not complete, points that should be there might not exist.
- ▶ The data only characterizes the complete metric spaces.

Data?

Definition

A computable metric space is a triple (X, d, α) where X is a set, d a metric on X and α a dense sequence in (X, d) such that...

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Quasi-Polish spaces, sales pitch

Quasi-Polish are the best setting for computable topology.

- ▶ Essentially everything works nicely for them.
- ▶ And if you try to generalize beyond, you often have to pay a price.

The reference



Matthew de Brecht:
Quasi-Polish spaces.
APAL, 2013

How to define with Quasi-Polish spaces

Quasi-Polish spaces are separable Smyth-complete quasi-metric spaces, but that is not useful.

How to define with Quasi-Polish spaces

Definition

Let \prec be a transitive relation on \mathbb{N} . A set $I \subseteq \mathbb{N}$ is an *ideal*, if

1. $I \neq \emptyset$
2. $n \in I$ and $m \prec n$ implies $m \in I$
3. $n, m \in I$ implies $\exists k \in I$ with $n \prec k$ and $m \prec k$.

By $I(\prec)$ we denote the space of ideals of \prec , represented by enumeration of their elements.

Definition

A space is pre-computably Quasi-Polish if it is isomorphic to $I(\prec)$ for some c.e. relation \prec . It is computably Quasi-Polish if it is additionally computably overt.

Hyperspace constructions

If \mathbf{X} is Quasi-Polish, then so are

1. $\mathcal{O}(\mathcal{O}(\mathbf{X}))$
2. $\mathcal{K}(\mathbf{X})$ and $\mathcal{V}(\mathbf{X})$
3. $\mathcal{P}(\mathbf{X})$ (the space of probability measures on \mathbf{X}).

But $\mathcal{O}(\mathbb{N}^{\mathbb{N}})$ is not Quasi-Polish (because it is not countably-based).

Contrast to “computable topological space”

Definition

A computable topological space is a triple $(X, (B_n)_{n \in \mathbb{N}}, R)$ where X is a set, $(B_n)_{n \in \mathbb{N}}$ the basis of a topology on X , and R a c.e. set such that

$$B_n \cap B_m = \bigcup_{\{k \mid (n,m,k) \in R\}} B_k$$

Only the set R is actual tangible data.

Theorem (de Brecht)

For every c.e. set R which is compatible with some computable topological space, there is a unique up to iso precomputably Quasi-Polish space S_R compatible with R such that every computable topological space compatible with R embeds into S_R .

A sense of completeness I

Theorem (de Brecht)

Every Quasi-Polish space has the property of Baire.

A sense of completeness II

Theorem (de Brecht, P. & Schröder)

A space \mathbf{X} is computably Quasi-Polish iff it admits a computably admissible effectively open total representation $\delta : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbf{X}$

Hyper spaces

Definition

For represented spaces \mathbf{X} , \mathbf{Y} , define the function space $\mathcal{C}(\mathbf{X}, \mathbf{Y})$ by $np \in \mathbb{N}^{\mathbb{N}}$ being a name of $f : X \rightarrow Y$, iff the n -th oracle machine with oracle p realizes f .

Definition (Sierpiński space)

Let \mathbb{S} be defined via $\delta_{\mathbb{S}}(0^{\mathbb{N}}) = \perp$ and $\delta_{\mathbb{S}}(p) = \top$ for $p \neq 0^{\mathbb{N}}$.

- ▶ In $\mathcal{O}(\mathbf{X})$, we provide just enough information about $U \subseteq \mathbf{X}$ to make $\in : \mathbf{X} \times \mathcal{O}(\mathbf{X}) \rightarrow \mathbb{S}$ computable.
- ▶ In $\mathcal{V}(\mathbf{X})$, we provide just enough information about $A \subseteq \mathbf{X}$ to make $\text{Intersects} : \mathcal{V}(\mathbf{X}) \times \mathcal{O}(\mathbf{X}) \rightarrow \mathbb{S}$ computable.

Overt subsets of $\{0, 1\}^{\mathbb{N}}$

Proposition

$A \subseteq \{0, 1\}^{\mathbb{N}}$ is computably overt iff $A = [T]$ for a c.e. pruned tree T .

Overt choice

Definition

Let $\text{VC}_{\mathbf{X}} : \subseteq \mathcal{V}(\mathbf{X}) \Rightarrow \mathbf{X}$ map $A \neq \emptyset$ to any $x \in A$.

- ▶ Very similar to closed choice.
- ▶ Often used implicitly to compute objects (such as crossing zeros of continuous functions)

Known results

Theorem (Brattka & Presser 2003)

If \mathbf{X} is a computable Polish space, then $\text{VC}_{\mathbf{X}}$ is computable.

Theorem (Brattka 2002)

$\text{VC}_{\mathbb{Q}}$ is not computable.

Basic result for quasi-Polish spaces

Theorem

Let \mathbf{X} be precomputably quasi-Polish. Then $\text{VC}_{\mathbf{X}}$ is computable.

Choquet-games

The *convergent strong Choquet game* is played as follows:

1. Player I first plays a pair (U_0, x_0) with $U_0 \in \mathcal{O}(\mathbf{X})$ and $x_0 \in U_0$.
2. Player II must respond with an open set V_0 such that $x_0 \in V_0 \subseteq U_0$.
3. Player I then responds with a pair (U_1, x_1) with U_1 open and $x_1 \in U_1 \subseteq V_0$.
4. then Player II must play an open V_1 with $x_1 \in V_1 \subseteq U_1$.
5. And so on.

Player II wins the game if and only if the sequence of opens $(V_i)_{i \in \mathbb{N}}$ is a neighborhood basis for a unique point in \mathbf{X} .

Theorem (de Brecht, Ruiyuan)

A non-empty countably based T_0 -space \mathbf{X} is quasi-Polish iff Player II has a winning strategy.

The connection

Theorem

Let \mathbf{X} be countably-based and T_1 . If $VC_{\mathbf{X}}$ is continuous, Player II has a winning strategy in the convergent strong Choquet game.

Corollary

A countably-base T_1 space \mathbf{X} is quasi-Polish iff $VC_{\mathbf{X}}$ is continuous.

Observation

$VC_{\mathbf{X}}$ can be computable even if \mathbf{X} is not precomputably quasi-Polish.

Weihrauch degrees of overt choice

Proposition

Let \mathbf{X} be effectively countably-based. Then ${}^1(\mathbf{VC}_{\mathbf{X}})$ is computable.

Introducing coPolish spaces

Theorem (Schröder)

Let \mathbf{X} be Hausdorff and admissible. The following are equivalent:

- 1. \mathbf{X} is a direct limit of compact metric spaces.*
- 2. The space $\mathcal{O}(\mathbf{X})$ is quasi-Polish and \mathbf{X} is regular.*
- 3. \mathbf{X} has an admissible representation with a locally compact domain.*
- 4. \mathbf{X} has a countable network consisting of compact subsets.*

Definition

Call those spaces coPolish.

Overview on the results

Theorem

For a CoPolish space \mathbf{X} exactly one of the following cases holds:

1. \mathbf{X} is Polish and $\text{VC}_{\mathbf{X}}$ is continuous.
2. \mathbf{X} is not countably-based, Fréchet-Urysohn, and $\text{ACC}_{\mathbb{N}} \leq_W^c \text{VC}_{\mathbf{X}} <_W^c \text{LPO}$.
3. \mathbf{X} is not Fréchet-Urysohn, and $\text{LPO} \leq_W^c \text{VC}_{\mathbf{X}} \leq_W^c \widehat{(\text{id} : \mathbb{S}_{\Pi_1^1} \rightarrow \mathbb{S})}$.

The references



Matthew de Brecht, Arno Pauly & Matthias Schröder:
Overt choice.
Computability, 2020



Mathieu Hoyrup:
Notes on Overt choice.
Computability, 2023

A philosophical question

Should we consider coPolish spaces to be complete?

A paradoxical theorem

Theorem (Kihara, Ng & P)

It is not the case that there are countably many effectively countably-based effectively Hausdorff spaces $(X_n)_{n \in \mathbb{N}}$ such that every effectively countably-based effectively Hausdorff space embeds into some X_n .

Separation notions as sparse-ness notions

Definition

Call $X \subseteq \mathbf{T}^\omega$ separated, if for any $x, y \in X$ with $x \neq y$ there is some n with $\{x(n), y(n)\} = \{0, 1\}$.

Observation

A space is effectively Hausdorff and effectively second-countable iff it is isomorphic to a separated subspace of \mathbf{T}^ω .

The reference



Takayuki Kihara, Keng Meng Ng & Arno Pauly:
Enumeration degrees and non-metrizable topology
Memoirs of the AMS, to appear

Definition

Definition (following Brattka & Gherardi)

Call a space \mathbf{X} multiretraceable, if there is a computable procedure that extends partial continuous maps

$f : \subseteq \{0, 1\}^{\mathbb{N}} \rightarrow \mathbf{X}$ to total continuous functions $\bar{f} : \{0, 1\}^{\mathbb{N}} \rightarrow \mathbf{X}$.

Definition (P. & Solda)

A completion of a space \mathbf{X} is an embedding $\iota : \mathbf{X} \rightarrow \bar{\mathbf{X}}$ into some multiretraceable space $\bar{\mathbf{X}}$.

Observation (Dzhafarov)

Every represented space has a completion.

Changing the base space for representations

Definition (Neumann, P., Pradic & Valenti)

Call a space \mathbf{X} \mathbf{Y} -represented, if there is a computable surjection $\delta : \mathbf{Y} \rightarrow \mathbf{X}$ such that $\delta^{-1} : \mathbf{X} \Rightarrow \mathbf{Y}$ is computable, too.

Proposition (NPPV)

If \mathbf{X} is $\omega^{\leq \omega}$ -represented, then it is multiretraceable.

Theorem (NPPV)

If \mathbf{X} is multiretraceable and either effectively countably based or of the form $\mathcal{O}(\mathbf{Z})$, then it is $\omega^{\leq \omega}$ -represented.

I am not aware of admissible multiretraceable spaces which are neither countably-based nor of the form $\mathcal{O}(\mathbf{X})$.

The reference



Mathieu Hoyrup:

The fixed-point property for represented spaces

APAL, 2022