Spaces with Unfixable Type

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Outline

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Semicomputable/computable sets

Let $Q = [0,1]^{\mathbb{N}}$ be the Hilbert cube endowed with the topology generated by rational balls $(B_i)_{i\in\mathbb{N}}$.

Definition

A compact set K in Q is

1. Semicomputable if the set

$$\{i \in \mathbb{N} : K \cap \overline{B}_i = \emptyset\}$$

   is c.e.

2. Computable if it is semicomputable and the set

$$\{i \in \mathbb{N} : K \cap B_i \neq \emptyset\}$$

   is c.e.

The line segment

Fact

One can build a segment $[a, 1]$ which is semicomputable but not computable.

Proof.

Create a real number $a = \sum_{n \in A} 2^{-n-1}$ where $A \subseteq \mathbb{N}$ is the halting set (a non-computable c.e. set). The segment $[a, 1]$ is semicomputable but not computable. □

Previous results

Some semicomputable sets are actually computable:

- The circle, and more generally n-dimensional spheres [Miller 2002],
- Closed manifolds [Iljazovic et al. 2013],
- ...

Computable type

Definition (Iljazovic)

A compact metrizable space X has computable type if every semicomputable copy of X in Q is computable.

Definition (Iljazovic)

A compact pair (X, A) has computable type if for every copy (Y, B) of (X, A) in Q, if (Y, B) is semicomputable then Y is computable.

Examples

For instance, the pair consisting of the line segment and its two endpoints has computable type.

And so does every finite graph and its endpoints.

A pair consisting of a compact manifold and its boundary has computable type.

Strong computable type

Definition (A., Hoyrup)

A compact metrizable space X has strong computable type if for every oracle O and every copy Y of X in Q, if Y is semicomputable relative to O, then Y is computable relative to O.

Definition (A., Hoyrup)

A compact pair (X, A) has strong computable type if for every oracle O and every copy (Y, B) of (X, A) in Q, if (Y, B) is semicomputable relative to O, then Y is computable relative to O.

The ϵ-surjection property

Definition (A., Hoyrup)

Let $\epsilon > 0$. A pair $(X, A) \subseteq Q$ satisfies the generalized ϵ-surjection property if every continuous function of pairs $f : (X, A) \to (X, A)$ satisfying $d_X(f, \mathrm{id}_X) < \epsilon$ is surjective.

Definition (A., Hoyrup)

The pair $(X, A) \subseteq Q$ satisfies the ϵ-surjection property if every continuous function $f : X \to X$ satisfying $d_X(f, \mathrm{id}_X) < \epsilon$ and $f|_A = \mathrm{id}_A$ is surjective.

X has the ϵ-surjection property if the pair $(X, \emptyset)$ does.

A necessary condition

Theorem (A., Hoyrup)

If $(X, A) \subseteq Q$ has strong computable type, then it satisfies the generalized ϵ-surjection property for some $\epsilon > 0$.

Outline

Unfixable type

Definition (A.)

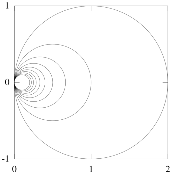
A compact metrizable space X has unfixable type, if for every compact subset $Y \subseteq X$, the pair (X, Y) does not have strong computable type.

There is no finite simplicial complex which has unfixable type.
There is no compact manifold which has unfixable type.

Compact infinite Wedge sum

Definition

Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of non-empty compact metrizable spaces none of them is a singleton. Let $(x_i)_{i\in\mathbb{N}}$ be a sequence of points such that for every i, $x_i \in X_i$. The compact Wedge sum $\vee_{i\in\mathbb{N}} X_i$ of $(X_i)_{i\in\mathbb{N}}$ with respect to $(x_i)_{i\in\mathbb{N}}$ is defined as follows. First, take the disjoint union of all the X_i's. Next, perform the one-point compactification by adding a point p. Finally, form the quotient space by identifying the points x_i (for each $i \in \mathbb{N}$) with p.



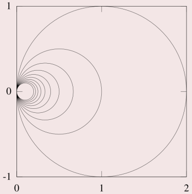
Compact infinite Wedge sums have unfixable type

Theorem (A.)

Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of non-empty compact metrizable spaces none of them is a singleton. Let $(x_i)_{i\in\mathbb{N}}$ be a sequence of points such that for every i, $x_i \in X_i$. The compact Wedge sum of $(X_i)_{i\in\mathbb{N}}$ with respect to $(x_i)_{i\in\mathbb{N}}$ has unfixable type.

Example

The Hawaiian earring has unfixable type.



Infinite products do not have strong computable type

Proposition (A.)

Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of non-empty compact metrizable spaces none of them is a singleton. The product $\Pi_{i\in\mathbb{N}}X_i$ does not have strong computable type.

Note that infinite products may have computable type.

The Hilbert cube and the Cantor space

Proposition (A.)

The Hilbert cube has unfixable type.

Proof.

Using the fact that the Hilbert cube is homogenuous and that n-cubes retract to proper subsets. □

Proposition (A.)

The Cantor space $2^{\mathbb{N}}$ has unfixable type.

Some questions

Question

- *Is unfixable type preserved by finite (countable) products?*
- *Do countable products of manifolds have unfixable type? In particular, does the countable product of circles have unfixable type?*
- *Do countable products of finite simplicial complexes have unfixable type?*
- *Can we characterize spaces with unfixable type?*

Outline

Optimal pairs

Definition (A.)

Let (X, A) be a compact pair which has strong computable type. (X, A) is optimal if for every compact set $B \subset A$, the pair (X, B) does not have strong computable type.

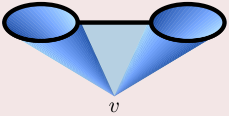
Example

Compact manifolds with boundary are optimal pairs.

Optimal pairs

Fact

Optimal pairs need not to exist.



Question

What are the compact pairs that have strong computable type and are optimal?

Thank you!