Infinite Loops

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Background

- Verification Community: consider some (discrete or continuous time) dynamical system.
- Computational question: does a given set of state reach some other set of states in finite time under the evolution of the system?
- Examples:
  - does a point reach a hyperplane under a matrix (Skolem Problem) – open,
  - does a point reach a half-space under a matrix (Positivity Problem) – open,
  - do all points in a half-space reach the complement of the half-space under a matrix – decidable under various interpretations of “all points”.
- The above examples all concern *linear systems*.
- Already for piecewise linear discrete updates it is undecidable whether some point/all points escape a half-space.

Background

- Computable Analysis is a very natural setting for systems verification:
  - Built-in notion of “robustness” which is important when system is not known exactly/to infinite precision.
  - Undecidability relies on difficult “corner cases” – we may get “better” decidability results, since we can only handle problem instances with a locally constant answer.
  - The set of instances with a locally constant answer may still be “large” in some sense.

Terminology

- Let X be some space, $S \subseteq X$ be some subset. Let $f : X \to X$ be some function.
- A point $x \in S$ is *trapped in* S *under* f if $f^n(x) \in S$ for all $n \in \mathbb{N}$.
- A point $x \in S$ *escapes* S *under* f if $f^n(x) \notin S$ for some $n \in \mathbb{N}$.
- We will study the “decidability” of two types of “escape problems” where $X = \mathbb{R}^d$, S is a closed/compact set, and f is a continuous map.
- Since verifying that x escapes is trivial if we have negative information on S, I will focus on detecting trapped points.
- This is only possible if we have positive information about the interior of S.

Detecting Trapped Points

Consider two Properties:

- A given point is trapped in a given open set:

$$\left\{(f,U,x_0)\in C(\mathbb{R}^d,\mathbb{R}^d)\times\mathscr{O}(\mathbb{R}^d)\times\mathbb{R}^d \mid \forall n. f^n(x_0)\in U\right\}$$

- There exists a trapped point in the given open set:

$$\left\{(f,U)\in C(\mathbb{R}^d,\mathbb{R}^d)\times\mathscr{O}(\mathbb{R}^d) \mid \exists x_0\in U. \forall n. f^n(x_0)\in U\right\}$$

Question. Are the interiors of these properties semi-decidable? A problem instance belonging to the interior of the set will also be called a *robust instance*.

Opens, Compacts, Overts

For a represented space X we define:

- $\mathscr{O}(X)$ = open subsets of X, identified with their characteristic function with values in Sierpinski space.
- $\mathscr{K}(X)$ = saturated compact sets K identified with the function
$$\{U \in \mathscr{O}(X) \mid K \subseteq U\}.$$
- $\mathscr{V}(X)$ = closed sets A identified with
$$\{U \in \mathscr{O}(X) \mid A \cap U \neq \emptyset\}.$$

Reminder

Trapped Point Problem:

$$\left\{(f, U, x_0) \in C(\mathbb{R}^d, \mathbb{R}^d) \times \mathscr{O}(\mathbb{R}^d) \times \mathbb{R}^d \mid \forall n. f^n(x_0) \in U\right\}$$

Robust Invariants

- A *robust invariant* for $f : \mathbb{R}^d \to \mathbb{R}^d$ is a pair $(V, K) \in \mathscr{O}(\mathbb{R}^d) \times \mathscr{K}(\mathbb{R}^d)$ with $V \subseteq K$ and $f(K) \subseteq V$.
- The set of all pairs $(V, K) \subseteq \mathscr{O}(\mathbb{R}^d) \times \mathscr{K}(\mathbb{R}^d)$ with $V \subseteq K$ is overt.
- The set of all pairs (V, K) with $f(K) \subseteq V$ is open.
- Hence, we can semi-decide if a robust invariant exists.
- For any $U \in \mathscr{O}(\mathbb{R}^d)$ we can semi-decide if there exists a robust invariant in U.

Trapped Point Problem

Lemma

An instance (f,U,x_0) *of the Trapped Point Problem is robust if and only if* f *has a robust invariant* (V,K) *with* $x_0 \in V \subseteq K \subseteq U$.

Corollary

The interior of the Trapped Point Problem is semi-decidable.

Proof of Lemma

- First assume that $\overline{U}$ is compact.
- We show that if f has no robust invariant (V,K) with $x_0 \in V \subseteq K \subseteq U$ then for all $\varepsilon > 0$ there exists $\widetilde{f}$ with $\left|\left|f - \widetilde{f}\right|\right|_{K,\infty} < \varepsilon$ such that x escapes U under $\widetilde{f}$.
- Let $E_0 = \mathbb{R}^d \setminus U$.
- Let $0 < \delta < \varepsilon/4$ be such that

  $$\forall x, y \in \overline{U}. d(x,y) < \delta \to d(f(x), f(y)) < \varepsilon/4$$

  and (w.l.o.g.) such that $x_0 \notin \overline{B}(E_0, \delta)$.
- Then $\overline{U} \setminus B(E_0,\delta)$ is not a robust invariant of f.
- Hence

  $$E_1 = \left\{x \in \overline{U} \setminus B(E_0,\delta) \mid f(x) \in \overline{B}(E_0,\delta)\right\} \neq \emptyset$$

Proof of Lemma

- Now, either $x_0 \in \overline{B}(E_1, \delta)$ or $\overline{U} \setminus (B(E_0,\delta) \cup B(E_1,\delta))$ cannot be a robust invariant either...
- In the second case we obtain a non-empty set

$$E_2 = \left\{ x \in \overline{U} \setminus (B(E_0,\delta) \cup B(E_1,\delta)) \mid f(x) \in \overline{B}(E_1,\delta) \right\} \neq \emptyset$$

- We obtain a finite (!) sequence of sets $E_0, E_1, \dots, E_N$ with
  1. $E_i \cap \bigcup_{j=0}^{i-1} B(E_j,\delta) = \emptyset$
  2. $x_0 \in B(E_N,\delta)$
  3. $f(x) \in \overline{B}(E_{i-1},\delta)$ for all $x \in E_i$
- Using these properties, we construct an ε-perturbation of f under which x escapes within N steps.

Proof of Lemma: The General Case

- The bounded open sets are dense in the open sets.
- If (f, U, x_0) is a robust instance, then (f, U', x_0) is a robust instance for some bounded open set.
- Hence f has a robust invariant in U' containing x_0.
- Hence f has a robust invariant in U containing x_0.

Remark

- In the construction for bounded U we have only perturbed f, not U or x_0.
- One could consider the special Trapped Point Problem for a fixed bounded U and x_0:
$$\left\{f : \mathbb{R}^d \to \mathbb{R}^d \mid \forall n. f^n(x_0) \in U\right\}.$$
- Our "generic" algorithm decides the interior of this.
- With a bit more work, the above can be extended all $U \neq \mathbb{R}^d$.
- Compare with the general situation:
$$\{(x,y) \in \mathbb{R} \mid x \in \mathbb{Q}\}.$$
Interior and interior of complement are empty. But for fixed $x \in \mathbb{R}$, the set
$$\{y \in \mathbb{R} \mid x \in \mathbb{Q}\}$$
is decidable.

Example

- If f admits a robust invariant (V, K) with $K \subseteq U$ and $f^n(x_0) \in V$, where $x_0, f(x_0), \dots, f^{n-1}(x_0) \subseteq U$, then f admits a robust invariant with (V', K') with $K' \subseteq U$ and $x_0 \in V'$.
- If f^n admits a robust invariant (V, K) with K for some n, then f admits a robust invariant (V', K') with $V \subseteq V'$ and $K \subseteq K'$.
- A $\mathscr{C}^1$-function f admits an *attracting cycle* if $f^n(x) = x$ for some x and n and
$$\left|\left|D(f^n)(x)\right|\right|_\infty < 1$$
- A point x_0 is *attracted* by an attracting cycle if for all $\varepsilon > 0$ there exists N such that $f^N(x_0)$ is ε-close to some point on the cycle.
- If x_0 does not escape and is attracted by an attracting cycle in U, then x_0 is contained in the interior of a robust invariant in U.

Example

- Mandelbrot set

$$\mathscr{M} = \{c \in \mathbb{C} \mid 0 \text{ is trapped in } B(0,3) \text{ under } f_c(z) = z^2 + c\}.$$

- It is unknown whether $\mathscr{M}$ is computable.
- Hertling: signed distance function of Mandelbrot set is computable subject to density of hyperbolicity conjecture.
- The only open problem/result that relies on the conjecture is semi-decidability of the interior of $\mathscr{M}$.
- A parameter $c \in \mathbb{C}$ is *hyperbolic* if f_c has an attracting cycle.
- Density of hyperbolicity conjecture: all parameters in M° are hyperbolic.
- Every attracting cycle attracts 0, so our algorithm semi-decides all hyperbolic parameters.
- One can show that for non-hyperbolic parameters, the corresponding Trapped Point Problem instance is not robust, so our algorithm semi-decides precisely the hyperbolic parameters.

“Size” of the Halting Set

- Our algorithm “generically decides” the Trapped Point Problem:
- The interior of the Trapped Point Problem is dense.

Some Point Trapped

- The "Some Point Trapped Problem":

$$\left\{(f,U)\in C(\mathbb{R}^d,\mathbb{R}^d)\times\mathscr{O}(\mathbb{R}^d)\mid \exists x_0\in U.\forall n.f^n(x_0)\in U\right\}$$

- Perhaps this "reduces" to the pointwise case?
- If we can semi-decide the interior of a set $A\subseteq X$, then we can lift this to algorithms for semi-deciding the interior of

$$\{K\in\mathscr{K}(X)\mid K\subseteq A\}$$

and of

$$\{V\in\mathscr{V}(X)\mid V\cap A\neq\emptyset\}$$

Some Point Trapped

- In particular, we can semi-decide the interiors of the sets

$$\left\{(f,U,K)\in C(\mathbb{R}^d,\mathbb{R}^d)\times\mathscr{O}(\mathbb{R}^d)\times\mathscr{K}(\mathbb{R}^d)\mid \forall x_0\in K.\forall n.f^n(x_0)\in U\right\}$$

  and

$$\left\{(f,U,V)\in C(\mathbb{R}^d,\mathbb{R}^d)\times\mathscr{O}(\mathbb{R}^d)\times\mathscr{V}(\mathbb{R}^d)\mid \exists x_0\in V.\forall n.f^n(x_0)\in U\right\}$$

- So in particular we can set $V = U$ in the above.
- But U is *open* and not overt!

Quantifying over Opens vs Overts

- Existentially quantifying over an overt set can be quite weak!
- Consider the set

$$\left\{(f,x)\in C(\mathbb{R})\times\mathbb{R} \mid f(x)=0\right\}.$$

  Interior = ∅.

- Consider the set

$$\left\{(f,V)\in C(\mathbb{R})\times\mathscr{V}(\mathbb{R}) \mid \exists x\in V.f(x)=0\right\}.$$

  Interior = ∅

- Now, consider the set

$$\left\{(f,U)\in C(\mathbb{R})\times\mathscr{O}(\mathbb{R}) \mid \exists x\in U.f(x)=0\right\}.$$

  Interior = $\left\{(f,U)\in C(\mathbb{R})\times\mathscr{O}(\mathbb{R}) \mid \exists x,y\in U.f(x)f(y)<0,[x,y]\subseteq U\right\}$.

Some Point Trapped

- There exist robust instances (f, U) of the Some Point Trapped Problem such that f has no robust invariant in U.
- Example: $U = B(0,1)$, $f = \text{rotation by } \theta \in (0, 2\pi)$.
- No individual point is robustly trapped under f.
- Yet, any small perturbation of f will have a fixed point near 0.

Robust Fixed Points and Robust Cycles

- We say that f has a robust fixed point in U if there exists a ball $B(x,r)$ with $\overline{B}(x,r) \subseteq U$ and an $\varepsilon > 0$ such that all $\widetilde{f}\colon \mathbb{R}^d \to \mathbb{R}^d$ with $||\widetilde{f} - f||_{\overline{B}(x,r),\infty} < \varepsilon$ have a fixed point in $B(x,r)$.
- More generally, we say that f has a *robust cycle* in U if f^n has a robust fixed point in U for some n.
- If f has a robust cycle in U, then (f,U) is a robust instance of the Some Point Trapped Problem.
- Can we computably detect this?

The Mapping Degree

- The d-sphere S^d has homology

$$H_n(S^d) \simeq \begin{cases} \mathbb{Z} & \text{if } n \in \{d, 0\} \\ 0 & \text{otherwise.} \end{cases}$$

- Any continuous map $f\colon S^d \to S^d$ induces a map $f_*\colon H_*(S^d) \to H_*(S^d)$ on homology.
- The map $f_d\colon H_d(S^d) \to H_d(S^d)$ is essentially a linear map $\mathbb{Z} \to \mathbb{Z}$, *i.e.* given by multiplication of a number.
- This number is called the *degree* of f.

The Mapping Degree

- Let $f\colon \mathbb{R}^d \to \mathbb{R}^d$ be continuous, let $B(x,r)$ be an open ball about x. Assume that $0 \notin f(\partial B(x,r))$. Then the map
$$\frac{f(x)}{||f(x)||}$$
can be viewed as a map $S^d \to S^d$ and assigned a degree $\deg(f, B(x,r))$.
- If $\deg(f, B(x,r)) \neq 0$, then the equation $f(x) = 0$ has a solution in $B(x,r)$.
- If $\deg(f, B(x,r)) = 0$, then for any $\varepsilon > 0$ there exists an ε-perturbation $\widetilde{f}$ of f such that the equation $\widetilde{f}(x) = 0$ does not have a solution in $B(x,r)$.
- Put in contrapositive, if the solution is stable under small perturbations of f, then the degree is non-zero.
- The degree can be uniformly computed using *computational homology*.

Detecting Robust Cycles

- The map f has a robust fixed point in U iff there exists a ball $B(x,r)$ with $\overline{B}(x,r) \subseteq U$, $0 \notin (f - \mathrm{id}_{\mathbb{R}^d})\left(\partial\overline{B}(x,r)\right)$ and $\deg(f - \mathrm{id}_{\mathbb{R}^d}, B(x,r)) \neq 0$.
- In particular, the existence of a robust cycle is semi-decidable.

Robust Cycles vs Robust Invariants

- There exists a map $f : \mathbb{R} \to \mathbb{R}$ with a robust fixed point in $(0,1)$ but no robust invariant.
- Let $A \subseteq \mathbb{R}^2$ be an annulus. There exists a map $f : A \to A$ with a robust invariant but without any cycles.
- Corollary: there exists a map $f : \mathbb{R}^3 \to \mathbb{R}^3$ with a robust invariant in $(0,1)^3$ but without any cycles in $(0,1)^3$.
- Open Question: Is there a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ with a robust invariant in $(0,1)^2$ but without any cycles in $(0,1)^2$?

The State of Some Point Trapped

- We can do two things simultaneously: search for a robust cycle and search for a robust invariant.
- Either one yields a dense open subset of the problem.
- Neither alone semi-decides the whole interior, except when $d = 1$, where searching for a robust cycle is complete (but searching for a robust invariant isn't).
- Open question: does the combination of the two criteria semi-decide the interior of the problem?