New definitions in the theory of Type 1 computable topological spaces

Emmanuel Rauzy

Alexander von HUMBOLDT STIFTUNG
Universität der Bundeswehr München

Seminar on computable topology, Kochel 2024

Outline

Marked groups

- A **marked group** (G, S) is a countable group G together with a finite generating family S.

Marked groups

- A **marked group** (G, S) is a countable group G together with a finite generating family S.
- A **relation** in a marked group (G, S) is a word over $S \cup S^{-1}$ evaluated as the identity in G.

Example

$\mathbb{Z}^2$ is generated by $a = (1, 0)$ and $b = (0, 1)$.

Example

$\mathbb{Z}^2$ is generated by $a = (1,0)$ and $b = (0,1)$.

$$ab^2ab^{-2}a^{-2} = (1,0) + (0,2) + (1,0) - (0,2) - (2,0) = (0,0) = 1.$$

Example

$\mathbb{Z}^2$ is generated by $a = (1, 0)$ and $b = (0, 1)$.

$$ab^2ab^{-2}a^{-2} = (1, 0) + (0, 2) + (1, 0) - (0, 2) - (2, 0) = (0, 0) = 1.$$

A relation in $(\mathbb{Z}^2, (a, b))$ is a word on a, b, a^{-1} and b^{-1} which contains as many a as a^{-1} and as many b as b^{-1}.

Key lemma

Lemma

A marked group is uniquely determined by the set of relations it satisfies.

- The set of words over $(S \cup S^{-1})$ can be identified with $\mathbb{N}$ via a standard bijection.

Representations of marked groups

- The set of words over $(S \cup S^{-1})$ can be identified with $\mathbb{N}$ via a standard bijection.
- Marked groups are then seen as subsets of $\mathbb{N}$, i.e. elements of $\mathcal{P}(\mathbb{N}) = \{0,1\}^{\mathbb{N}}$.

Representations of marked groups

- The set of words over $(S \cup S^{-1})$ can be identified with $\mathbb{N}$ via a standard bijection.
- Marked groups are then seen as subsets of $\mathbb{N}$, i.e. elements of $\mathcal{P}(\mathbb{N}) = \{0,1\}^{\mathbb{N}}$.
- The **space of marked groups** is the set of elements of $\{0,1\}^{\mathbb{N}}$ that define a marked group with the topology induced by the Cantor space topology.

Representations of marked groups

- The set of words over $(S \cup S^{-1})$ can be identified with $\mathbb{N}$ via a standard bijection.
- Marked groups are then seen as subsets of $\mathbb{N}$, i.e. elements of $\mathcal{P}(\mathbb{N}) = \{0,1\}^{\mathbb{N}}$.
- The **space of marked groups** is the set of elements of $\{0,1\}^{\mathbb{N}}$ that define a marked group with the topology induced by the Cantor space topology.
- It thus equipped the representation induced by that of the Cantor space.

Banach-Mazur computable functions

(X, ρ) and (Y, τ) represented sets.

Banach-Mazur computable functions

(X, ρ) and (Y, τ) represented sets.

Definition

A function $f : (X, \rho) \to (Y, \tau)$ is **Banach-Mazur computable** it maps ρ-computable sequences to τ-computable sequences.

Motivating theorem

Take P a group property.

Motivating theorem

Take P a group property.

Theorem (R.)

The following are equivalent:

- *There exists a finitely presented group G with solvable word problem where the problem of determining if a finitely generated subgroup of G has P is not semi-decidable;*
- *P is not Banach-Mazur semi-decidable.*

Motivating theorem

Take P a group property.

Theorem (R.)

The following are equivalent:

- *There exists a finitely presented group G with solvable word problem where the problem of determining if a finitely generated subgroup of G has P is not semi-decidable;*
- *P is not Banach-Mazur semi-decidable.*

Theorem (Folklore)

Finitely presented groups are exactly the fundamental groups of closed manifolds (compact and without boundary).

Remark

Theorem (Reformulation of Boone-Rogers)

The space of marked groups is a Π^0_1-subset of $\{0,1\}^{\mathbb{N}}$ which is not overt, nor computably separable.

Some problems

- Study Markov and Banach-Mazur continuity theorems and try to apply them to the space of marked groups.

Some problems

- Study Markov and Banach-Mazur continuity theorems and try to apply them to the space of marked groups.
- Give a rigorous framework to study Type 1 and Banach-Mazur computability on non-computably separable metric spaces.

Some problems

- Study Markov and Banach-Mazur continuity theorems and try to apply them to the space of marked groups.
- Give a rigorous framework to study Type 1 and Banach-Mazur computability on non-computably separable metric spaces.
- Study effective second countability (and non-effective second countability).

Some problems

- Give a rigorous framework to study Type 1 and Banach-Mazur computability on non-computably separable metric spaces.

Outline

Three goals

- Giving a general notion of **Type 1 computable topological space;**

Three goals

- Giving a general notion of **Type 1 computable topological space;**
- Type 1 notion of computable basis that works for **uncountable** spaces as long as computable points are dense;

Three goals

- Giving a general notion of **Type 1 computable topological space;**
- Type 1 notion of computable basis that works for **uncountable** spaces as long as computable points are dense;
- General definition of **the computable topology generated by a computable metric** that does not rely on effective separability.

A dubious goal and a reasonable goal

- Type 1 notion of computable basis that works for uncountable spaces as long as computable points are dense.

A dubious goal and a reasonable goal

- Type 1 notion of computable basis that works for uncountable spaces as long as computable points are dense.
  - Is this really interesting?
  - What is the point?

A dubious goal and a reasonable goal

- Type 1 notion of computable basis that works for uncountable spaces as long as computable points are dense.
    - Is this really interesting?
    - What is the point?
- General definition of **the computable topology generated by a computable metric** that does not rely on effective separability.

A dubious goal and a reasonable goal

- Type 1 notion of computable basis that works for uncountable spaces as long as computable points are dense.
  - Is this really interesting?
  - What is the point?
- General definition of **the computable topology generated by a computable metric** that does not rely on effective separability.
  - This is really interesting, not only for Type 1 computability.

A dubious goal and a reasonable goal

- Type 1 notion of computable basis that works for uncountable spaces as long as computable points are dense.
  - Is this really interesting?
  - What is the point?
- General definition of **the computable topology generated by a computable metric** that does not rely on effective separability.
  - This is really interesting, not only for Type 1 computability.
  - See work of Bauer and Lešnik in synthetic topology or the Schröder Metrization theorem.

A dubious goal and a reasonable goal

- Type 1 notion of computable basis that works for uncountable spaces as long as computable points are dense.
  - Is this really interesting?
  - What is the point?
- General definition of **the computable topology generated by a computable metric** that does not rely on effective separability.
  - This is really interesting, not only for Type 1 computability.
  - See work of Bauer and Lešnik in synthetic topology or the Schröder Metrization theorem.
- The first point naturally provides the answer to the second one -so it must be somewhat interesting.

Outline

Type 1 computability

A numbering of a set X is a partial surjection $\nu :\subseteq \mathbb{N} \to X$.

Type 1 computability

A numbering of a set X is a partial surjection $\nu :\subseteq \mathbb{N} \to X$.
If $\nu(n) = x$, then n is a ν-name of x.

Type 1 computability

A numbering of a set X is a partial surjection $\nu :\subseteq \mathbb{N} \to X$.
If $\nu(n) = x$, then n is a ν-name of x.

A function f between numbered sets (X, ν) and (Y, μ) is computable if it is possible, given the ν-name of a point x in X, to compute a μ-name for its image $f(x)$.

Type 1 computable topological space

Definition

A *Type 1 computable topological space* is a quadruple $(X, \nu, \mathcal{T}_c, \tau)$ where $\nu :\subseteq \mathbb{N} \to X$ is a numbering of X, $\mathcal{T}_c$ is a subset of $\mathcal{P}(X)$, $\tau :\subseteq \mathbb{N} \to \mathcal{T}_c$ is a numbering of $\mathcal{T}_c$, and such that:

Type 1 computable topological space

Definition

A *Type 1 computable topological space* is a quadruple $(X, \nu, \mathcal{T}_c, \tau)$ where $\nu :\subseteq \mathbb{N} \to X$ is a numbering of X, $\mathcal{T}_c$ is a subset of $\mathcal{P}(X)$, $\tau :\subseteq \mathbb{N} \to \mathcal{T}_c$ is a numbering of $\mathcal{T}_c$, and such that:

1. The empty set and X both belong to $\mathcal{T}_c$;

Type 1 computable topological space

Definition

A *Type 1 computable topological space* is a quadruple $(X, \nu, \mathcal{T}_c, \tau)$ where $\nu :\subseteq \mathbb{N} \to X$ is a numbering of X, $\mathcal{T}_c$ is a subset of $\mathcal{P}(X)$, $\tau :\subseteq \mathbb{N} \to \mathcal{T}_c$ is a numbering of $\mathcal{T}_c$, and such that:

1. The empty set and X both belong to $\mathcal{T}_c$;
2. The open sets in the image of τ are uniformly ν-semi-decidable;

Type 1 computable topological space

Definition

A *Type 1 computable topological space* is a quadruple $(X, \nu, \mathcal{T}_c, \tau)$ where $\nu :\subseteq \mathbb{N} \to X$ is a numbering of X, $\mathcal{T}_c$ is a subset of $\mathcal{P}(X)$, $\tau :\subseteq \mathbb{N} \to \mathcal{T}_c$ is a numbering of $\mathcal{T}_c$, and such that:

1. The empty set and X both belong to $\mathcal{T}_c$;
2. The open sets in the image of τ are uniformly ν-semi-decidable;
3. The operations of taking computable unions and finite intersections are computable for τ.

Outline

Recursive metric spaces

A *Type 1 computable metric space* is a triple (X, ν, d), where ν is a partial numbering of X and $d : X \times X \to \mathbb{R}$ is a ν-computable metric.

Recursive metric spaces

A *Type 1 computable metric space* is a triple (X, ν, d), where ν is a partial numbering of X and $d : X \times X \to \mathbb{R}$ is a ν-computable metric.
(Given n, m and q, we can compute $d(\nu(n), \nu(m))$ within 2^{-q}.)

Notions of computable topology on a RMS: Lacombe approach

We can try to define a computable topology on (X, ν, d) by saying that the computably open sets should be the **computable unions of open balls.**

Notions of computable topology on a RMS: Lacombe approach

We can try to define a computable topology on (X, ν, d) by saying that the computably open sets should be the **computable unions of open balls.**

Let $W_i = \mathrm{dom}(\varphi_i)$, and $c_\mathbb{R}$ the Cauchy numbering of computable reals. We define τ by

$$\tau(i) = \bigcup_{\langle n,m\rangle \in W_i} B(\nu(n), c_\mathbb{R}(m)).$$

Notions of computable topology on a RMS: Lacombe approach

We can try to define a computable topology on (X, ν, d) by saying that the computably open sets should be the **computable unions of open balls.**

Let $W_i = \mathrm{dom}(\varphi_i)$, and $c_{\mathbb{R}}$ the Cauchy numbering of computable reals. We define τ by

$$\tau(i) = \bigcup_{\langle n,m\rangle \in W_i} B(\nu(n), c_{\mathbb{R}}(m)).$$

This is the Lacombe approach. (Lacombe, 1957)

Notions of computable topology on a RMS: Lacombe approach

We can try to define a computable topology on (X, ν, d) by saying that the computably open sets should be the **computable unions of open balls.**

Let $W_i = \text{dom}(\varphi_i)$, and $c_{\mathbb{R}}$ the Cauchy numbering of computable reals. We define τ by

$$\tau(i) = \bigcup_{\langle n,m\rangle \in W_i} B(\nu(n), c_{\mathbb{R}}(m)).$$

This is the Lacombe approach. (Lacombe, 1957)

It does not work on non computably separable metric spaces.

Problem with Lacombe approach

Indeed, for the Lacombe approach to work, we have to be able to uniformly express the intersection of two open balls as a computable union of open balls.

Problem with Lacombe approach

Indeed, for the Lacombe approach to work, we have to be able to uniformly express the intersection of two open balls as a computable union of open balls.

- Overtness may fail.
- Computable open choice may fail.

Problem with Lacombe approach

Indeed, for the Lacombe approach to work, we have to be able to uniformly express the intersection of two open balls as a computable union of open balls.

- Overtness may fail.
- Computable open choice may fail.

There is a Type 1 computable metric space (X, ν, d) where the following version of computable choice fails:

- Input: two open balls B_1 and B_2 that intersect.
- Output: the name of a point in their intersection.

Notions of computable topology on a RMS: Nogina approach, 1966

Definition

Let (X, ν, d) be a Type 1 computable metric space.
Define a numbering τ by the following: $\tau(\langle n, m\rangle) = O$ iff

- n encodes O as a ν-semi-decidable set;
- m encodes a program that, on input the ν-name of a point x in O, produces the name of a computable real r such that $B(x, r) \subseteq O$.

Moschovakis' Theorem, 1964

Theorem

On a Type 1 computably Polish space, the Nogina and Lacombe approaches agree.

Summary

	Metric	+Effective separability	+Effective completeness
Nogina	✔	✔	✔
Lacombe	×	✔	

The problem with metric spaces

The problem is thus that there is not a single computable topology that we can call “the effective topology induced by a computable metric”.

Left computable reals

Denote by $c_{\nearrow} :\subseteq \mathbb{N} \to \mathbb{R}_{\nearrow}$ the numbering associated to left computable reals.

Description of the Metric topology

Let (X, ν, d) be a Type 1 computable metric space. We define a computable topology $(\mathcal{T}_c, \tau)$:

Description of the Metric topology

Let (X, ν, d) be a Type 1 computable metric space. We define a computable topology $(\mathcal{T}_c, \tau)$:
The τ-name of an open set O is an encoded pair (n, m),

- n gives the code of O as a semi-decidable set,
- m encodes a $(\nu, c_{\nearrow})$-computable function $F : O \to \mathbb{R}^+_{\nearrow}$, which satisfies the following:

$$\forall x \in O,\ B(x, F(x)) \subseteq O;$$

$$\forall x \in O,\ \exists r > 0,\ \forall y \in B(x, r),\ F(y) > r.$$

Description of the Metric topology

Let (X, ν, d) be a Type 1 computable metric space. We define a computable topology $(\mathcal{T}_c, \tau)$:
The τ-name of an open set O is an encoded pair (n, m),

- n gives the code of O as a semi-decidable set,
- m encodes a $(\nu, c_{\nearrow})$-computable function $F : O \to \mathbb{R}^+_{\nearrow}$, which satisfies the following:

$$\forall x \in O,\ B(x, F(x)) \subseteq O;$$

$$\forall x \in O,\ \exists r > 0,\ \forall y \in B(x, r),\ F(y) > r.$$

Theorem (R.)

This indeed defines a Type 1 computable topology.

Problem solved

Theorem (New Moschovakis-type theorem)

On a computably separable Type 1 computable metric space, the computable topology defined above agrees with the Lacombe topology.

Summary

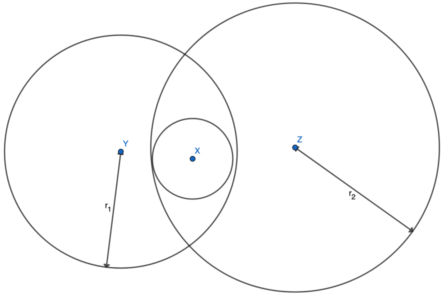
<table>
<tr><th></th><th>Metric</th><th>+Effective separability</th><th>+Effective completeness</th></tr>
<tr><td>Nogina</td><td>✔</td><td>✔</td><td rowspan="3">✔</td></tr>
<tr><td>$\rightarrow$</td><td>✔</td><td rowspan="2">✔</td></tr>
<tr><td>Lacombe</td><td>$\times$</td></tr>
</table>

Outline

What have we used from metric spaces

In a metric space, we have a function that shows that open balls form a basis:

$$I(x, B(y, r_1), B(z, r_2)) = \min(r_1 - d(x, y), r_2 - d(x, z)).$$



What have we used from metric spaces

In a metric space, we have a function that shows that open balls form a basis:

$$I(x, B(y, r_1), B(z, r_2)) = \min(r_1 - d(x, y), r_2 - d(x, z)).$$

The function I satisfies two interesting properties:

What have we used from metric spaces

In a metric space, we have a function that shows that open balls form a basis:

$$I(x, B(y, r_1), B(z, r_2)) = \min(r_1 - d(x, y), r_2 - d(x, z)).$$

The function I satisfies two interesting properties:

- $I(x, B(y, r_1), B(z, r_2)) = \min(r_1 - d(x, y), r_2 - d(x, z))$ is increasing in r_1 and r_2, and it does not depend on the given name of x, y and z.

What have we used from metric spaces

In a metric space, we have a function that shows that open balls form a basis:

$$I(x, B(y, r_1), B(z, r_2)) = \min(r_1 - d(x, y), r_2 - d(x, z)).$$

The function I satisfies two interesting properties:

- $I(x, B(y, r_1), B(z, r_2)) = \min(r_1 - d(x, y), r_2 - d(x, z))$ is increasing in r_1 and r_2, and it does not depend on the given name of x, y and z.
- The function I does not vanish inside $B(y, r_1) \cap B(z, r_2)$: for any x in $B(y, r_1) \cap B(z, r_2)$, there is a radius I such that for any point p in $B(x, I)$, x belong to $B(p, \min(r_1 - d(y, p), r_2 - d(z, p)))$.

Formal inclusion relations

Used as early as 1981 by Klaus Weihrauch and Ulrich Schreiber.

Formal inclusion relations

Used as early as 1981 by Klaus Weihrauch and Ulrich Schreiber.

Definition (Spreen, 1998)

Let $\mathfrak{B}$ be a subset of $\mathcal{P}(X)$, and β a numbering of $\mathfrak{B}$. Let $\mathring{\subseteq}$ be a binary relation on $\mathrm{dom}(\beta)$. We say that $\mathring{\subseteq}$ is a **strong inclusion relation** *for* $(\mathfrak{B}, \beta)$ if the following hold:

- The relation $\mathring{\subseteq}$ is transitive;
- $\forall n, m \in \mathrm{dom}(\beta),\ n \mathring{\subseteq} m \implies \beta(n) \subseteq \beta(m)$.

Formal inclusion relations

Used as early as 1981 by Klaus Weihrauch and Ulrich Schreiber.

Definition (Spreen, 1998)

Let $\mathfrak{B}$ be a subset of $\mathcal{P}(X)$, and β a numbering of $\mathfrak{B}$. Let $\mathring{\subseteq}$ be a binary relation on $\mathrm{dom}(\beta)$. We say that $\mathring{\subseteq}$ is a **strong inclusion relation** *for* $(\mathfrak{B}, \beta)$ if the following hold:

- The relation $\mathring{\subseteq}$ is transitive;
- $\forall n, m \in \mathrm{dom}(\beta)$, $n \mathring{\subseteq} m \implies \beta(n) \subseteq \beta(m)$.

I restrict my attention to reflexive strong inclusion relations, I call them **formal inclusion relations**.

Formal inclusion relations

Used as early as 1981 by Klaus Weihrauch and Ulrich Schreiber.

Definition (Spreen, 1998)

Let $\mathfrak{B}$ be a subset of $\mathcal{P}(X)$, and β a numbering of $\mathfrak{B}$. Let $\mathring{\subseteq}$ be a binary relation on $\mathrm{dom}(\beta)$. We say that $\mathring{\subseteq}$ is a **strong inclusion relation** *for* $(\mathfrak{B}, \beta)$ if the following hold:

- The relation $\mathring{\subseteq}$ is transitive;
- $\forall n, m \in \mathrm{dom}(\beta),\ n \mathring{\subseteq} m \implies \beta(n) \subseteq \beta(m)$.

I restrict my attention to reflexive strong inclusion relations, I call them **formal inclusion relations**.

Let $\mathring{\sim}$ be the equivalence relation induced by $\mathring{\subseteq}$.

The metric topology in terms of formal inclusion relations

$$I(x, B(y, r_1), B(z, r_2)) = \min(r_1 - d(x, y), r_2 - d(x, z)).$$

The metric topology in terms of formal inclusion relations

$$I(x, B(y, r_1), B(z, r_2)) = \min(r_1 - d(x, y), r_2 - d(x, z)).$$

The function I satisfies:

- $I(x, B(y, r_1), B(z, r_2)) = \min(r_1 - d(x, y), r_2 - d(x, z))$ is increasing in r_1 and r_2, and it does not depend on the given name of x, y and z. $\boxed{\mathring{\sim}}$
- The function I does not vanish inside $B(y, r_1) \cap B(z, r_2)$: for any x in $B(y, r_1) \cap B(z, r_2)$, there is a radius I such that for any point p in $B(x, I)$, x belong to $B(p, \min(r_1 - d(y, p), r_2 - d(z, p)))$. $\circledcirc$

The metric topology in terms of formal inclusion relations

In general, $I(x, B(y, r_1), B(z, r_2))$ is replaced by $I(n, b_1, b_2)$, $n \in \mathrm{dom}(\nu)$ and $b_1, b_2 \in \mathrm{dom}(\beta)$.

The metric topology in terms of formal inclusion relations

In general, $I(x, B(y, r_1), B(z, r_2))$ is replaced by $I(n, b_1, b_2)$, $n \in \mathrm{dom}(\nu)$ and $b_1, b_2 \in \mathrm{dom}(\beta)$.

In terms of formal inclusion relation:

$$\boxed{\mathring{\sim}}\ \forall n, m \in \mathrm{dom}(\nu),\ \nu(n) = \nu(m) \implies \forall b_1 \mathring{\subseteq} \hat{b}_1, b_2 \mathring{\subseteq} \hat{b}_2 \in \mathrm{dom}(\beta),\ \implies I(n, b_1, b_2) \mathring{\subseteq} I(m, \hat{b}_1, \hat{b}_2);$$

The metric topology in terms of formal inclusion relations

In general, $I(x, B(y, r_1), B(z, r_2))$ is replaced by $I(n, b_1, b_2)$, $n \in \mathrm{dom}(\nu)$ and $b_1, b_2 \in \mathrm{dom}(\beta)$.

In terms of formal inclusion relation:

$$\boxed{\mathring{\sim}}\ \forall n, m \in \mathrm{dom}(\nu),\ \nu(n) = \nu(m) \implies$$
$$\forall b_1 \mathring{\subseteq} \hat{b}_1, b_2 \mathring{\subseteq} \hat{b}_2 \in \mathrm{dom}(\beta), \implies I(n, b_1, b_2) \mathring{\subseteq} I(m, \hat{b}_1, \hat{b}_2);$$

$$\circledcirc\ \forall b_1, b_2 \in \mathrm{dom}(\beta), \forall x \in \beta(b_1) \cap \beta(b_2) \implies \exists b_3 \in \mathrm{dom}(\beta),$$
$$x \in \beta(b_3) \,\&\, \forall m \in \mathrm{dom}(\nu),\ \nu(m) \in \beta(b_3) \implies b_3 \mathring{\subseteq} I(m, b_1, b_2).$$

New notion of basis

Definition

A **Spreen topological basis** *for* (X, ν) is a triple $(\mathfrak{B}, \beta, \mathring{\subseteq})$, where $(\mathfrak{B}, \beta)$ is a numbered basis with a formal inclusion relation $\mathring{\subseteq}$, that satisfies the following conditions:

1. The elements of the basis $(\mathfrak{B}, \beta)$ are uniformly semi-decidable;
2. There is a computable function $T : \mathrm{dom}(\nu) \to \mathrm{dom}(\beta)^{\mathbb{N}}$ which testifies for the fact that X itself will be computably open;
3. There is a computable function $I : \mathrm{dom}(\nu) \times \mathrm{dom}(\beta) \times \mathrm{dom}(\beta) \to \mathrm{dom}(\beta)^{\mathbb{N}}$ which allows to compute intersections.

Conditions on T

The function $T : \mathrm{dom}(\nu) \to \mathrm{dom}(\beta)^{\mathbb{N}}$ should satisfy the following:

$$\forall n \in \mathrm{dom}(\nu),\ T(n) = (u_k)_{k\in\mathbb{N}} \implies \forall k \in \mathbb{N},\ \nu(n) \in \beta(u_k);$$

$$\boxed{\mathring{\sim}}\ \forall n, m \in \mathrm{dom}(\nu),\ \nu(n) = \nu(m) \implies T(n) \mathring{\sim} T(m);$$

$$\circledcirc\ \forall x \in X,\ \exists b \in \mathrm{dom}(\beta),\ x \in \beta(b)\ \&\ \forall n \in \mathrm{dom}(\nu),$$
$$\nu(n) \in \beta(b) \implies b \mathring{\subseteq} T(n).$$

Conditions on I

The function $I : \text{dom}(\nu) \times \text{dom}(\beta) \times \text{dom}(\beta) \to \text{dom}(\beta)^{\mathbb{N}}$ should satisfy the following.

Conditions on I

The function $I : \text{dom}(\nu) \times \text{dom}(\beta) \times \text{dom}(\beta) \to \text{dom}(\beta)^{\mathbb{N}}$ should satisfy the following.
For any n in dom(ν), b_1 and b_2 in dom(β), with $\nu(n) \in \beta(b_1) \cap \beta(b_2)$, we have $I(n, b_1, b_2) = (u_k)_{k \in \mathbb{N}}$, with:

$$\forall k \in \mathbb{N},\ \nu(n) \in \beta(u_k);$$

$$\forall k \in \mathbb{N},\ \beta(u_k) \subseteq \beta(b_1)\,\&\,\beta(u_k) \subseteq \beta(b_2);$$

Conditions on I

The function $I : \mathrm{dom}(\nu) \times \mathrm{dom}(\beta) \times \mathrm{dom}(\beta) \to \mathrm{dom}(\beta)^{\mathbb{N}}$ should satisfy the following.
For any n in $\mathrm{dom}(\nu)$, b_1 and b_2 in $\mathrm{dom}(\beta)$, with $\nu(n) \in \beta(b_1) \cap \beta(b_2)$, we have $I(n, b_1, b_2) = (u_k)_{k\in\mathbb{N}}$, with:

$$\forall k \in \mathbb{N},\ \nu(n) \in \beta(u_k);$$

$$\forall k \in \mathbb{N},\ \beta(u_k) \subseteq \beta(b_1)\,\&\,\beta(u_k) \subseteq \beta(b_2);$$

$$\boxed{\mathring{\sim}}\ \forall n, m \in \mathrm{dom}(\nu),\ \nu(n) = \nu(m) \implies$$
$$\forall b_1 \mathring{\subseteq} \hat{b}_1,\ b_2 \mathring{\subseteq} \hat{b}_2 \in \mathrm{dom}\ (\beta),\ \implies I(n, b_1, b_2) \mathring{\subseteq} I(m, \hat{b}_1, \hat{b}_2);$$

Conditions on I

The function $I : \mathrm{dom}(\nu) \times \mathrm{dom}(\beta) \times \mathrm{dom}(\beta) \to \mathrm{dom}(\beta)^{\mathbb{N}}$ should satisfy the following.
For any n in $\mathrm{dom}(\nu)$, b_1 and b_2 in $\mathrm{dom}(\beta)$, with $\nu(n) \in \beta(b_1) \cap \beta(b_2)$, we have $I(n, b_1, b_2) = (u_k)_{k\in\mathbb{N}}$, with:

$$\forall k \in \mathbb{N},\ \nu(n) \in \beta(u_k);$$

$$\forall k \in \mathbb{N},\ \beta(u_k) \subseteq \beta(b_1)\ \&\ \beta(u_k) \subseteq \beta(b_2);$$

$\boxed{\mathring{\sim}}$ $\forall n, m \in \mathrm{dom}(\nu),\ \nu(n) = \nu(m) \implies$

$$\forall b_1 \mathring{\subseteq} \hat{b}_1, b_2 \mathring{\subseteq} \hat{b}_2 \in \mathrm{dom}\ (\beta),\ \implies I(n, b_1, b_2) \mathring{\subseteq} I(m, \hat{b}_1, \hat{b}_2);$$

$\circledcirc$ $\forall b_1, b_2 \in \mathrm{dom}(\beta),\ \forall x \in \beta(n_1) \cap \beta(n_2) \implies \exists b_3 \in \mathrm{dom}(\beta),$
$x \in \beta(b_3)\ \&\ \forall m \in \mathrm{dom}(\nu),\ \nu(m) \in \beta(b_3) \implies b_3 \mathring{\subseteq} I(m, b_1, b_2).$

Spreen topology

Definition

Let $(X, \nu, \mathcal{B}, \beta, \mathring{\subseteq})$ be a numbered set equipped with a Spreen basis. The computable topology associated to the Spreen basis $(\mathcal{B}, \beta)$ is given by a numbering τ, which is defined as follows:

Spreen topology

Definition

Let $(X, \nu, \mathcal{B}, \beta, \mathring{\subseteq})$ be a numbered set equipped with a Spreen basis. The computable topology associated to the Spreen basis $(\mathcal{B}, \beta)$ is given by a numbering τ, which is defined as follows: Put $\tau(\langle n, m\rangle) = O$ if and only if:

1. n is a name of O given as a semi-decidable set;

Spreen topology

Definition

Let $(X, \nu, \mathcal{B}, \beta, \mathring{\subseteq})$ be a numbered set equipped with a Spreen basis. The computable topology associated to the Spreen basis $(\mathcal{B}, \beta)$ is given by a numbering τ, which is defined as follows:
Put $\tau(\langle n, m\rangle) = O$ if and only if:

1. n is a name of O given as a semi-decidable set;
2. m codes a computable function $F : \nu^{-1}(O) \to \mathrm{dom}(\beta)^{\mathbb{N}}$ such that:

$$\forall n \in \mathrm{dom}(\nu),\ \nu(n) \in O \,\&\, F(n) = (u_k)_{k\in\mathbb{N}}$$
$$\implies \forall k \in \mathbb{N},\ \nu(n) \in \beta(u_k) \,\&\, \beta(u_k) \subseteq O.$$

Conditions $\boxed{\mathring{\sim}}$ and $\circledcirc$ for F

Additionally, F should satisfy:

$$\boxed{\mathring{\sim}}\ \forall n, m \in \mathrm{dom}(F),\ \nu(n) = \nu(m) \implies F(n) \mathring{\sim} F(m);$$

Conditions $\boxed{\mathring{\sim}}$ and $\circledcirc$ for F

Additionally, F should satisfy:

$$\boxed{\mathring{\sim}}\ \forall n, m \in \mathrm{dom}(F),\ \nu(n) = \nu(m) \implies F(n) \mathring{\sim} F(m);$$

$$\circledcirc\ \forall x \in O,\ \exists b \in \mathrm{dom}(\beta),\ x \in \beta(b)\ \&$$
$$\forall n \in \mathrm{dom}(\nu),\ \nu(n) \in \beta(b) \implies b \mathring{\subseteq} F(n).$$

Main result on Spreen bases

Theorem (R.)

The computable topology associated to a Spreen basis is a computable topology.

Moschovakis' Theorem on Lacombe sets is trivialized

Theorem (Moschovakis' theorem on Lacombe sets for Spreen bases)

Let $(\mathfrak{B}, \beta, \mathring{\subseteq})$ be a Spreen basis on (X, ν). If there exists a ν-computable sequence which is dense for the topology generated by $\mathfrak{B}$, then the computable Spreen topology is computably equivalent to the Lacombe topology, where open sets are given as computable unions of basic sets.

Outline

Open problems

- Continuity problem on the space of marked group for Markov or Banach-Mazur computability;
- Continuity problem for marked groups given by co-enumerations of relations.

Open problems

- Continuity problem on the space of marked group for Markov or Banach-Mazur computability;
- Continuity problem for marked groups given by co-enumerations of relations.
- An embedding $X \hookrightarrow (Y, \rho)$ is *intrinsic* we have an equivalence of representations $[\rho_{|X} \to \mathbb{S}] \equiv [\rho \to \mathbb{S}] \cap X$. Explain why non-intrinsic embeddings occur more in Type 1 than in Type 2 computability.

Open problems

- Continuity problem on the space of marked group for Markov or Banach-Mazur computability;
- Continuity problem for marked groups given by co-enumerations of relations.
- An embedding $X \hookrightarrow (Y, \rho)$ is *intrinsic* we have an equivalence of representations $[\rho_{|X} \to \mathbb{S}] \equiv [\rho \to \mathbb{S}] \cap X$. Explain why non-intrinsic embeddings occur more in Type 1 than in Type 2 computability.
- Does the space of marked groups has a representation that makes it computably Polish?
- Let X be a second countable space. Is it possible that every standard Weihrauch representation makes it not computably separable, while a representation that makes it computably separable is not equivalent to a standard Weihrauch representation?

Je vous remercie pour votre attention

Thank you for your attention