Principal Spaces

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Survey

- Definition
- Examples
- Bizarre Properties
- Open Questions

Definition

Let X be a topological space.

- $\mathcal{F} \subseteq \mathrm{O}(X)$ is called a *filter* on the lattice $\mathrm{O}(X)$, if
    - $X \in \mathcal{F}$,
    - $\mathcal{F}$ is proper: $\emptyset \notin \mathcal{F}$
    - $\mathcal{F}$ is upwards-closed: $\mathcal{F} \ni U \subseteq V \in \mathrm{O}(X) \implies V \in \mathcal{F}$
    - $\mathcal{F}$ is closed under finite intersections.
- A *prime filter* $\mathcal{F}$ is a filter such that
  $$U \cup V \in \mathcal{F} \implies U \in \mathcal{F} \text{ or } V \in \mathcal{F}$$
- An *ultrafilter* $\mathcal{F}$ is a maximal filter, i.e. it cannot be extended to another proper filter.

Definition

Filters, prime filters and ultrafilters on $\mathrm{A}(X)$ are defined similarly (without swapping the roles of $\cap, \cup$!)

Remarks

- $\mathcal{N}(x) := \{U \text{ open} \mid x \in U\}$ is called the *neighbourhood filter* of $x \in X$.
- $\mathcal{N}(x)$ is a prime filter.
- Filters generated by a point are called *principal*.
- The dual filter $\mathcal{N}^\sharp(x) := \{A \text{ closed} \mid x \in A\}$ on $\mathrm{A}(X)$ is
  - also prime
  - even an ultrafilter, whenever X is T_1.

Definition

We call a topological space X *principal*, if for any prime filter $\mathcal{F}$ on $\mathrm{O}(X)$ there is a unique $x \in X$ such that $\mathcal{F} = \mathcal{N}(x)$.

Remark

- X is *sober*, if for any *completely prime* filter $\mathcal{F}$ on $\mathrm{O}(X)$ there is a unique $x \in X$ such that $\mathcal{F} = \mathcal{N}(x)$.
- $\mathcal{F}$ is *completely prime*, if
  $$\bigcup(\mathcal{U}) \in \mathcal{F} \implies \exists U \in \mathcal{U}. U \in \mathcal{F} \text{ for any } \mathcal{U} \subseteq \mathrm{O}(X).$$
- Any Hausdorff space is sober.
- If X carries an effectively admissible representation, then the map $\mathcal{N}(x) \mapsto x$ is computable.

Fact

Every principal space is sober and T_0.

Example (Principal spaces)

- Every finite T_0-space.
- The sobrification of the cofinite topology on $\mathbb{N}$.
- [M. de Brecht]
  Any countably-based sober Noetherian space.
    - *Noetherian space:* every open subspace is compact.

Non-Example

- $P(\mathbb{N})$ with the Scott-topology.
- $O(X)$ with the Scott-topology for any infinite T_1-space X.

Proposition

$\mathbb{N}$ is not principal in ZFC.

Proof

- Zorn's Lemma entails the existence of a "free" ultrafilter $\mathcal{U}$ on $P(\mathbb{N})$.
- "Free" means that $\bigcap(\mathcal{U}) = \emptyset$.
- Ultrafilters are prime.
- The prime filter $\mathcal{U}$ is not equal to $\mathcal{N}(n)$ for any $n \in \mathbb{N}$, because $n \in \bigcap(\mathcal{N}(n))$.

Principal spaces in ZFC

Proposition

No infinite Hausdorff space is principal in ZFC.

This can be deduced from the non-principality of $\mathbb{N}$ *and:*

Lemma

Principal T_1-spaces are closed under:

- subspaces
- sequentialisation ($=$ sequential coreflection)

An alternative axiomatic setting

Weak Ultrafilter Axiom WUF:

There exists an ultrafilter $\mathcal{U}$ on the subsets of $\mathbb{N}$ that is *free*,

i.e. $\bigcap(\mathcal{U}) = \emptyset$

Recall

- ZF+DC+WUF $\Longrightarrow$ $\mathbb{N}$ is not principal
- ZFC $\Longrightarrow$ WUF

Proposition (ZF+DC)

If there is any infinite principal Hausdorff space, then $\neg$WUF.

The Axiom $\neg$WUF

For any ultrafilter $\mathcal{U}$ on the subsets of $\mathbb{N}$ there is some $x \in \bigcap(\mathcal{U})$.

Proposition

$\mathbb{N}$ is principal in ZF+DC+$\neg$WUF.

Proof

- Any prime filter $\mathcal{F}$ on $P(\mathbb{N})$ is an ultrafilter.
- By $\neg$WUF there is some $x \in \mathbb{N}$ with $x \in \bigcap(\mathcal{F})$.
- So $\mathcal{F} \subseteq \mathcal{N}(x)$.
- Now let $M \in \mathcal{N}(x)$.
- Primality and $\mathbb{N} = M \cup \complement M$ imply either $M \in \mathcal{F}$ or $\complement M \in \mathcal{F}$.
- Since $x \notin \complement M$ and $\mathcal{F} \subseteq \mathcal{N}(x)$, we have $M \in \mathcal{F}$.
- Altogether: $\mathcal{F} = \mathcal{N}(x)$.

Theorem (cf. [Schechter 1997])

- Shelah's model ZF+DC+BP entails ¬WUF.
- ZF+DC+AD entails ¬WUF.
- ZF, ZFC and ZF+DC+BP are equiconsistent.

Remember

- DC: Axiom of Dependent Choice
- BP: Baire Property Axiom
- AD: Axiom of Determinancy '

Conclusion

We can safely work in ZF+DC+¬WUF.

Theorem

In ZF+DC+¬WUF, every functionally Hausdorff QCB-space is principal.

Remember

- *Functionally Hausdorff:* *(= completely Hausdorff)* two points $x \neq y$ can be "separated" by a continuous real-valued function s with $s(x) \neq s(y)$.
- *QCB-space*: a **q**uotient of a **c**ountably **b**ased top. space.
- QCB: class of top. spaces that can be handled by TTE.

Proposition

The category of functionally Hausdorff QCB-spaces

- is cartesian-closed
- has all countable limits and all countable co-products.

Example

In ZF+DC+¬WUF, the following spaces are principal:

- all separable metric spaces, e.g.:
  - $\mathbb{R}$
  - $\mathbb{N}$
  - $\mathbb{N}^{\mathbb{N}}$
- Copolish spaces
- all locally convex vector spaces handable in TTE, e.g.:
  - $\mathbb{R}^k$
  - the ℓ_p-spaces equipped with the weak* topology
  - the space Poly of polynomials
  - the space $\mathcal{D}$ of test functions
  - the space $\mathcal{D}'$ of distributions
  - the space $\mathcal{A}$ of analytic functions

Proposition

The one-point compactification of any principal Hausdorff space is principal.

Example

In ZF+DC+$\neg$WUF, the sobrification of the Grünhage-Streicher space is principal.

Definition

We call X *semi-principal*, if every prime filter $\mathcal{F}$ on $\mathrm{A}(X)$ satisfies $\bigcap(\mathcal{F}) \neq \emptyset$.

Proposition

$X \in \mathsf{Top}$ is principal iff X is sober and every open subspace is semi-principal.

Proposition

A Hausdorff space X is principal iff X and $X \setminus \{z\}$ are semi-principal for every $z \in X$.

Proposition

In ZF+DC+¬WUF, principal T_1-spaces are closed under:

- countable products
- countable coproducts

Counterexample

- $2^{\mathbb{N}}$ is not principal in ZF+DC+WUF.
- $\mathrm{Si}^{\mathbb{N}}$ is not principal in ZF+DC.
- $\mathbb{N} \equiv \bigoplus_{i\in\mathbb{N}} 1$ is not principal in ZF+DC+WUF.

Proposition

In ZF+DC, principal spaces are closed under

- finite products
- finite coproducts

Bizarre Properties of Principal Spaces

Proposition

Let Y be principal and $X \in \mathsf{Top}$.
Let $h\colon \mathrm{O}(Y) \to \mathrm{O}(X)$ be a function that preserves

- binary intersection and binary union.

Then:

- h is Scott-continuous.
- h preserves arbitrary unions.

This resembles:

Theorem ([J.D.M. Wright 1977])

Let f be a *linear* function from a Banach space to a Hausdorff topological vector space. Then f is continuous in ZF+DC+BP.

Proposition

Let Y be principal and $X \in \mathsf{Top}$.
Let $h\colon \mathrm{O}(Y) \to \mathrm{O}(X)$ preserve $\cap, \cup, \emptyset, \top$.
Then there is a (continuous) function $f\colon X \to Y$ satisfying

$$f^{-1}[V] = h(V) \quad \text{for all open } V \subseteq Y.$$

Proposition

Let Y be principal and $X \in \mathsf{Top}$.
Let $h\colon \mathrm{O}(Y) \to \mathrm{O}(X)$ preserve $\cap, \cup$.
Then there is a partial (continuous) function $f\colon X \dashrightarrow Y$ satisfying

- $\mathrm{dom}(f) = h(Y) \setminus h(\emptyset)$
- $f^{-1}[V] = h(V) \cap \mathrm{dom}(f) \quad$ for all open $V \subseteq Y$.

In the following:

Let X be a T_6-space ($=$ every closed set is functionally closed).

Definition

- An *ideal* on the commutative ring $\mathrm{C}(X, \mathbb{R})$ is a subset $\mathcal{I} \subseteq \mathrm{C}(X, \mathbb{R})$ such that
  - $\vec{0} \in \mathcal{I}$.
  - $\mathcal{I}$ is proper: $\mathcal{I} \neq \mathrm{C}(X, \mathbb{R})$.
  - $\mathcal{I}$ is closed under addition.
  - $f \in \mathcal{I}$ and $r \in \mathrm{C}(X, \mathbb{R})$ implies $f \cdot r \in \mathcal{I}$.
- An ideal $\mathcal{I}$ is *prime*, if $f \cdot g \in \mathcal{I}$ implies $f \in \mathcal{I}$ or $g \in \mathcal{I}$.
- An ideal $\mathcal{I}$ is *maximal*, if it cannot be extended to a strictly larger proper ideal.

Definition

- For an ideal $\mathcal{I}$ on $\mathrm{C}(X, \mathbb{R})$ let
  $$\mathcal{Z}(\mathcal{I}) := \left\{ f^{-1}\{0\} \,\middle|\, f \in \mathcal{I} \right\}.$$
- For a filter $\mathcal{F}$ on $\mathrm{A}(X)$ let
  $$\Delta(\mathcal{F}) := \left\{ f \in \mathrm{C}(X, \mathbb{R}) \,\middle|\, f^{-1}\{0\} \in \mathcal{F} \right\}.$$

Lemma

- $\mathcal{Z}$ maps prime ideals to prime filters.
- Δ maps prime filters to prime ideals.

Lemma (cf. [Engelking 1989])

- $\mathcal{Z}$ maps maximal ideals to ultrafilters.
- Δ maps ultrafilters to maximal ideals.
- Maximal ideals and ultrafilters are in a bijective relationship via $\mathcal{Z}, \Delta$.

Proposition

Let X be a principal T_6-space.
Let $\mathcal{I}$ be a prime ideal on $\mathrm{C}(X, \mathbb{R})$.

- There is a (unique) $z \in X$ such that $f(z) = 0$ for all $f \in \mathcal{I}$.
- There is some $g \in \mathcal{I}$ which has z as its unique zero.

Remark

By contrast, in ZFC each infinite T_6-space admits an ideal $\mathcal{I}$ such that there is no common zero of all functions in $\mathcal{I}$.

Corollary

Let X be a principal T_6-space.
Every prime ideal $\mathcal{I}$ can be extended to a maximal ideal.

Proof

Namely $\left\{ f \in \mathrm{C}(X, \mathbb{R}) \,\middle|\, f(z) = 0 \right\}$ for the above z.

Remark

In ZFC, every ideal lies in a maximal ideal.

Open Questions

Recall

Given a principal T_6-space X, for any ideal $\mathcal{I}$ there is z with

$$\mathcal{I} \subseteq \left\{ f \in \mathrm{C}(X, \mathbb{R}) \,\middle|\, f(z) = 0 \right\}.$$

Open Questions

- Does this z satify even $\mathcal{I} = \left\{ f \in \mathrm{C}(X, \mathbb{R}) \,\middle|\, f(z) = 0 \right\}$?
- Is the map $\mathcal{I} \mapsto z$ computable ?

Proposition

Let Z be a principal *zero-dimensional* qcb-space.
Then for every prime ideal $\mathcal{I}$ there is some $z \in Z$ such that

$$\mathcal{I} = \left\{ f \in \mathrm{C}(Z, \mathbb{R}) \,\middle|\, f(z) = 0 \right\}$$

Open Questions

Consider ZF+DC+¬WUF (or ZF+DC+BP).

- Is every Hausdorff QCB-space principal?
- Is there any non-separable metric space that is principal?
- Is every principal space separable?
- Is every principal space Lindelöf?
- Give a characterisation of principal spaces.

Summary

- Principality is a generalisation of soberness.
- In ZFC, principal spaces are very rare.
- By contrast, in Shelah's alternative model ZF+DC+BP there are plenty of them.
- Examples are all functionally Hausdorff qcb-spaces.
- Principal topological spaces have some bizarre properties.

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Proposition

Let X be a topological space such that

- X is separable and functionally Hausdorff
- $X \times X$ is hereditarily Lindelöf.

Then X is principal in ZF+DC+¬WUF.