Type 1 Computability versus Type 2 Computability for Functions and for Relations, Counterexamples

Peter Hertling

Fakultät für Informatik, Universität der Bundeswehr München,
Neubiberg, Germany

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Overview

- Representations and Type 2 Computability,
  Numberings and Type 1 Computability
- For Relations between Computable Elements:
  Type 2 Computable $\Rightarrow$ Type 1 Computable
- A Converse for dcpo's:
  Extension of the Myhill/Shepherdson Theorem
- A Converse for Computable Metric Spaces:
  Extension of Tseitin's Theorem
- Counterexamples for Functions
- Counterexamples for Relations

Numberings

- A *numbering* of a set X is a surjective function $\nu :\subseteq \mathbb{N} \to X$. We call a pair (X, ν) consisting of a set X and a numbering ν of X a *numbered set*.
- Let (X, ν_X) and (Y, ν_Y) be numbered sets. We say that a function $F :\subseteq \mathbb{N} \to \mathbb{N}$ is a (ν_X, ν_Y)*-realizer* of a relation $R \subseteq X \times Y$ if, for all $n \in \mathrm{dom}(\nu_X)$ with $\nu_X(n) \in \mathrm{dom}(R)$, $n \in \mathrm{dom}(F)$ and $F(n) \in \mathrm{dom}(\nu_Y)$ and $\nu_Y(F(n)) \in R[\{\nu_X(n)\}]$.
- We say that a relation $R \subseteq X \times Y$ is (ν_X, ν_Y)*-computable* (*Type 1 computable*) if there exists a computable (ν_X, ν_Y)-realizer of R.
- If ν and ν' are numberings of one and the same set X then we say that ν *can be reduced to* ν' and write $\nu \leq \nu'$ if the identity function $\mathrm{id}_X : X \to X$ is (ν, ν')-computable.
- We call such ν and ν' *equivalent* and write $\nu \equiv \nu'$ if $\nu \leq \nu'$ and $\nu' \leq \nu$.
- If $\nu_X \equiv \nu'_X$ and $\nu_Y \equiv \nu'_Y$, then a relation is (ν_X, ν_Y)-computable iff it is (ν'_X, ν'_Y)-computable.

Representations

- A *representation* of a set X is a surjective function $\delta :\subseteq \mathbb{B} \to X$. We call a pair (X, δ) consisting of a set X and a representation δ of X a *represented set*.
- Let (X, δ_X) and (Y, δ_Y) be represented sets. We say that a function $F :\subseteq \mathbb{B} \to \mathbb{B}$ is a (δ_X, δ_Y)*-realizer* of a relation $R \subseteq X \times Y$ if, for all $p \in \mathrm{dom}(\delta_X)$ with $\delta_X(p) \in \mathrm{dom}(R)$, $p \in \mathrm{dom}(F)$ and $F(p) \in \mathrm{dom}(\delta_Y)$ and $\delta_Y(F(p)) \in R[\{\delta_X(n)\}]$.
- We say that a relation $R \subseteq X \times Y$ is (δ_X, δ_Y)*-computable* (*Type 2 computable*) if there exists a computable (δ_X, δ_Y)-realizer of R.
- If δ and δ' are representations of one and the same set X then we say that δ *can be reduced to* δ' and write $\delta \leq \delta'$ if the identity function $\mathrm{id}_X : X \to X$ is (δ, δ')-computable.
- We call such δ and δ' *equivalent* and write $\delta \equiv \delta'$ if $\delta \leq \delta'$ and $\delta' \leq \delta$.
- If $\delta_X \equiv \delta'_X$ and $\delta_Y \equiv \delta'_Y$, then a relation is (δ_X, δ_Y)-computable iff it is (δ'_X, δ'_Y)-computable.

Representations, Computable Elements, and a Numbering of them

- Let $P^{(1)}$ be the set of computable functions $f :\subseteq \mathbb{N} \to \mathbb{N}$, let $R^{(1)}$ be the set of computable functions $f : \mathbb{N} \to \mathbb{N}$.
- Let $\varphi : \mathbb{N} \to P^{(1)}$ be a standard numbering of $P^{(1)}$, $\varphi' := \varphi|^{R^{(1)}}$.

Let (X, δ_X) be a represented set.

- We call an element $x \in X$ δ_X*-computable* if there exists a computable $p \in R^{(1)} \cap \mathrm{dom}(\delta_X)$ with $\delta_X(p) = x$.
- If γ_X is a representation of X with $\gamma_X \equiv \delta_X$ then x is δ_X-computable iff x is γ_X-computable.
- Let X^{c} be the set of δ_X-computable elements of X.
- The function $\delta_X \circ \varphi'$ is a numbering of X^{c}.
- Furthermore, if γ_X is a representation of X that is equivalent to δ_X then $\gamma_X \circ \varphi'$ is equivalent to $\delta_X \circ \varphi'$.

Type 2 Computability implies Type 1 Computability

Theorem

Let (X, δ_X) and (Y, δ_Y) be represented sets. Let $R \subseteq X \times Y$ be a relation on $X \times Y$. Let $R^c := R \cap (X^c \times Y^c)$. If R is (δ_X, δ_Y)-computable then

$$\mathrm{dom}(R^c) = \mathrm{dom}(R) \cap X^c$$

and R^c is $(\delta_X \circ \varphi', \delta_Y \circ \varphi')$-computable.

In particular, if a relation $R \subseteq X^c \times Y^c$ is (δ_X, δ_Y)-computable then it is $(\delta_X \circ \varphi', \delta_Y \circ \varphi')$-computable.

Question: Under which circumstances is the converse true?

Effective Topological Spaces

- Let us call a triple (X, τ, B) an *effective topological space* if
  1. X is a set,
  2. τ is a T_0-topology on X that has a countable basis,
  3. $B : \mathbb{N} \to \tau$ is a total numbering of a basis of τ.
- Kreitz and Weihrauch (1985) have shown that the following function $\delta_B :\subseteq \mathbb{B} \to X$ is a representation of X with useful properties:

$$\begin{aligned} \mathrm{dom}(\delta_B) &:= \{p \in \mathbb{B} \mid (\exists x \in X)\mathrm{En}(p) = \{i \in \mathbb{N} \mid x \in B_i\}\}, \\ \delta_B(p) &:= \text{the unique } x \in X \text{ with } \mathrm{En}(p) = \{i \in \mathbb{N} \mid x \in B_i\}, \end{aligned}$$

  for $p \in \mathrm{dom}(\delta_B)$.
- Any representation of X equivalent to δ_B will be called a *standard representation of X*.
- Any numbering of X^c equivalent to the numbering $\nu_B := \delta_B \circ \varphi'$ of X^c will be called a *standard numbering of X^c*.

A Topological Consequence of Type 1 Computability

- If τ is a topology on a set X then for $x, y \in X$ we write $x \leq_\tau y$ if $\{U \in \tau \mid x \in U\} \subseteq \{U \in \tau \mid y \in U\}$.

Theorem

Let (X, σ, B) and (Y, τ, C) be effective topological spaces, and let $\gamma :\subseteq \mathbb{B} \to X$ and $\delta :\subseteq \mathbb{B} \to Y$ be standard representations of X respectively of Y. Let $R \subseteq X^{\mathrm{c}} \times Y^{\mathrm{c}}$ be a $(\gamma \circ \varphi', \delta \circ \varphi')$-computable relation. Then for all elements $x, y \in \mathrm{dom}(R)$ with $x \leq_\sigma y$ and all $i \in \mathbb{N}$: if $R[\{x\}] \subseteq C_i$ then $R[\{y\}] \cap C_i \neq \emptyset$.

Corollary

Let $f :\subseteq X^{\mathrm{c}} \to Y^{\mathrm{c}}$ be a $(\gamma \circ \varphi', \delta \circ \varphi')$-computable function. Then for any elements $x, y \in \mathrm{dom}(f)$ with $x \leq_\sigma y$ one obtains $f(x) \leq_\tau f(y)$.

Effective Continuity

- Let (X, σ, B) and (Y, τ, C) be effective topological spaces. A function $f :\subseteq X \to Y$ is called *effectively continuous* if there exists a c.e. set $A \subseteq \mathbb{N}$ such that, for all $j \in \mathbb{N}$,

$$f^{-1}[C_j] = \mathrm{dom}(f) \cap \bigcup \{B_i \mid \langle i, j \rangle \in A\}.$$

Lemma

Let (X, σ, B) and (Y, τ, C) be effective topological spaces. If a function $f :\subseteq X \to Y$ is effectively continuous then it is Type 2 computable.

- We call an effective topological space (X, σ, B) a *semi-computable* topological space if there exists a c.e. set $A \subseteq \mathbb{N}$ such that for all $i, j \in \mathbb{N}$, $B_i \cap B_j = \bigcup_{\langle k,i,j \rangle \in A} B_k$.

Theorem

Let (X, σ, B) be a semi-computable topological space, and let (Y, τ, C) be an effective topological space. Then a function $f :\subseteq X \to Y$ is effectively continuous iff it is Type 2 computable.

Dcpo's and the Scott topology

Let $(Z, \leq)$ be a partially ordered set.

- A subset $S \subseteq Z$ is called *directed* if it is nonempty and for any two elements $x, y \in S$ there exists an upper bound $z \in S$ of the set $\{x, y\}$.
- The poset $(Z, \leq)$ is called a *dcpo* if for any directed subset $S \subseteq Z$ there exists a supremum of S in Z.
- The set of all subsets $U \subseteq Z$ satisfying the following two conditions:
  1. U is upwards closed,
  2. for every directed subset $S \subseteq Z$ such that $\sup(S)$ exists and $\sup(S) \in U$ the intersection $S \cap U$ is not empty,

  is a T_0 topology on Z, called the *Scott topology*.

Continuous and Effective dcpo's

- We say that an element $x \in Z$ *approximates* an element $y \in Z$ and write $x \ll y$ if for all directed subsets $D \subseteq Z$ with $y \leq \sup(D)$ there exists an element $d \in D$ with $x \leq d$.
- We say that a subset $B \subseteq Z$ is a $\leq$*-basis* of Z if for every element $x \in Z$ the set

$$B_x := \{b \in B \mid b \ll x\}$$

  is directed and satisfies $\sup(B_x) = x$.
- A dcpo $(Z, \leq)$ is called *continuous* if there exists a $\leq$-basis in it.
- We call a triple $(Z, \leq, \beta)$ an *effective dcpo* if
  - $(Z, \leq)$ is a continuous dcpo,
  - $\beta : \mathbb{N} \to Z$ is a function such that the set $\{\beta(i) \mid i \in \mathbb{N}\}$ is a $\leq$-basis of $(Z, \leq)$.

Lemma

Let $(Z, \leq, \beta)$ be an effective dcpo. Let $B : \mathbb{N} \to \mathcal{P}(Z)$ be defined by

$$B(i) := \{z \in Z \mid \beta(i) \ll z\}.$$

Then $(Z, \sigma(Z, \leq), B)$ is an effective topological space.

An Extension of the Myhill/Shepherdson Theorem (1955)

- We call a triple $(Z, \leq, \beta)$ a *computable dcpo* if it is an effective dcpo and the set

$$\{\langle i, j \rangle \mid \beta(i) \ll \beta(j)\}$$

is computably enumerable.

Theorem (Spreen/Young 1984)

Let $(X, \leq, \beta)$ be a computable dcpo, let $\sigma := \sigma(X, \leq)$ be the Scott topology on X. Let (Y, τ, C) be an effective topological space. Then for a total function $f : X^{\mathrm{c}} \to Y^{\mathrm{c}}$ the following three conditions are equivalent:

1. *f is Type-1-computable,*
2. *f is effectively continuous,*
3. *f is Type-2-computable.*

(Semi-)(Computable) Metric Spaces

- A *metric space with a numbering of a basis (MNB)* is a triple (M, d, α) such that (M, d) is a metric space and $\alpha : \mathbb{N} \to M$ is a function such that the set $\alpha[\mathbb{N}]$ is a dense subset of M.
- An MNB (M, d, α) is called *semi-computable* if the set $D^\alpha_< := \{(i, j, q) \in \mathbb{N}^2 \times \mathbb{Q} \mid d(\alpha(i), \alpha(j)) < q\}$ is computably enumerable.
- An MNB (M, d, α) is called *computable* if the two sets $D^\alpha_<$ and $D^\alpha_> := \{(i, j, q) \in \mathbb{N}^2 \times \mathbb{Q} \mid d(\alpha(i), \alpha(j)) > q\}$ are computably enumerable.
- For an MNB (M, d, α) we define $B^\alpha : \mathbb{N} \to \mathcal{P}(M)$ by

$$B^\alpha\langle i, k\rangle := B(\alpha(i), 2^{-k}) = \{y \in M \mid d(\alpha(i), y) < 2^{-k}\}.$$

Lemma

1. *If (M, d, α) is an MNB then the triple (M, τ, B^α), where τ is the topology on M induced by the metric d, is an effective topological space.*
2. *If (M, d, α) is a semi-computable metric space then the triple (M, τ, B^α) is a semi-computable topological space.*

Computable Separability and Type 1 Computability

- Let (X, σ, B) be an effective topological space. A subset $Y \subseteq X$ is called *computably separable* if there is a c.e. set $A \subseteq \mathbb{N}$ such that $A \subseteq \nu_B$ and $\nu_B[A]$ is a dense subset of Y.

Lemma

Let (X, σ, B) be an effective topological space, and let (Y, d, β) be a computable metric space. Let us consider a function $f :\subseteq X^{\mathrm{c}} \to Y^{\mathrm{c}}$.

1. *f is Type 1 computable iff there exists a relation $R \subseteq (X^{\mathrm{c}} \times \mathbb{N}) \times Y$ with the following three properties:*
   I. *It is Type 1 computable.*
   II. *For every $n \in \mathbb{N}$ and $x \in \mathrm{dom}(f)$, $((x, n), f(x)) \in R$.*
   III. *For every $(x, n) \in \mathrm{dom}(R)$, $\mathrm{diam}(R[\{(x, n)\}]) \leq 2 \cdot 2^{-n}$.*
2. *If f is Type 1 computable and its domain $\mathrm{dom}(f)$ is computably separable then there exists a relation $R \subseteq (X \times \mathbb{N}) \times Y$ with I, II, III and with:*
   IV. *$\mathrm{dom}(R)$ is computably separable.*

An Extension of Tseitin's Theorem (1959/62) I (special case: Kreisel/Lacombe/Shoenfield (1959))

Let (X, σ, B) be an effective topological space, and let (Y, d, β) be a computable metric space. For a function $f :\subseteq X^{c} \to Y^{c}$ the following three conditions

(a) f is Type 1 computable and $\mathrm{dom}(f)$ is computably separable,

(b) there exists a relation $R \subseteq (X^{c} \times \mathbb{N}) \times Y$ with I, II, III, IV,

(c) f is Type 1 computable,

satisfy the implications:

$$(a) \underset{\not\Leftarrow}{\Longrightarrow} (b) \text{ and } (b) \underset{\not\Leftarrow}{\Longrightarrow} (c).$$

Theorem (H.)

Let (X, σ, B) be an effective topological space, and let (Y, d, β) be a computable metric space. If for a function $f :\subseteq X^{c} \to Y^{c}$ there exists a relation $R \subseteq (X^{c} \times \mathbb{N}) \times Y$ with I, II, III, IV, then f is Type 2 computable.

An Extension of Tseitin's Theorem II

Theorem (H.)

Let (X, d, α) and (Y, d, β) be computable metric spaces. Then for a function $f :\subseteq X^c \to Y^c$ the following three conditions are equivalent:

1. *f is effectively continuous,*
2. *f is Type 2 computable,*
3. *there exists a relation $R \subseteq (X^c \times \mathbb{N}) \times Y$ with I, II, III, IV.*

Counterexamples for Functions I

Proposition (Friedberg 1958, Spreen 2001)

Let $X = \mathbb{B}$ be the Baire space (then X is a computable metric space and $X^c = R^{(1)}$). Let $Y = \mathbb{S}$ be the Sierpinski space (then Y is a computable dcpo, and $Y^c = \mathbb{S}$). There exists a Type 1 computable function $f : R^{(1)} \to \mathbb{S}$ that is discontinuous and, therefore, neither effectively continuous nor Type 2 computable.

Counterexamples for Functions II

Proposition (Myhill 1959)

There exists a Type 1 computable function $f :\subseteq R^{(1)} \to \{0, 1\}$ *that is discontinuous and, therefore, neither effectively continuous nor Type 2 computable.*

Theorem (H. 1996)

Let $Y = \mathbb{B}$ *be the Baire space (then* Y *is a computable metric space and* $Y^{\mathrm{c}} = R^{(1)}$*). There exists a function* $f :\subseteq \mathbb{N} \to Y^{\mathrm{c}}$ *that is Type 2 computable (and effectively continuous, and there exists a relation* $R \subseteq (X^{\mathrm{c}} \times \mathbb{N}) \times Y$ *with I, II, III, IV, and it is Type 1 computable), but such that there is no Type 1 computable extension* $\tilde{f}$ *of* f *with computably separable domain.*

The Case of Relations

- Spreen (2010) has considered the question

  when Type 1 computability implies effective continuity

  in the general case when R is a relation, not just a function (for various effective continuity notions for relations).
- He has formulated conditions that ensure that a Type-1-computable relation is effectively continuous (in various senses).
- We present some counterexamples: total Type 1 computable relations (defined on all computable elements) that are not Type 2 computable, in fact, not even Type 2 continuous (that means: there is no continuous Type 2 realizer).

Counterexamples for Relations: Total on a Computable Metric Space

- A relation $R \subseteq X \times Y$ is called *total* if $\mathrm{dom}\, R := \{x \in X \mid (\exists y \in Y)(x, y) \in R\} = X$.

Theorem

1. *There exists a total Type 1 computable relation* $R \subseteq \mathbb{R}_c \times \{0, 1\}$ *that is not Type 2 continuous.*
2. *There exists a total Type 1 computable relation* $R \subseteq \mathbb{R}_c \times \mathbb{R}_c$ *that is not Type 2 continuous.*

Counterexamples for Relations: Total on a Computable dcpo

- The interval $[0, 1]$ with the usual $\leq$-relation is a complete lattice, hence, a dcpo as well. It is in a natural way a computable dcpo.
- The computable elements of this dcpo are the left-computable numbers in $[0, 1]$. A real number is called *left-computable* if there exists a computable, nondecreasing sequence of rational numbers converging to it.
- Let $\mathbb{R}_{\mathrm{lc}}$ be the set of left-computable real numbers, and set $[0, 1]_{\mathrm{lc}} := [0, 1] \cap \mathbb{R}_{\mathrm{lc}}$.

Theorem

There exists a total Type 1 computable relation $R \subseteq [0, 1]_{\mathrm{lc}} \times [0, 1]_{\mathrm{lc}}$ that is not Type 2 continuous.