Infinite Loops

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Compositional products and diamonds

- We denote by

$$f \star g$$

the compositional product by two problems f, g.

- For $n \in \mathbb{N}$ we denote by

$$f^{[n]} := \underbrace{f \star \ldots \star f}_{n-\text{times}}.$$

the n–fold compositional product of f by itself.

- Neumann and Pauly (2018) introduced the diamond

$$f \mapsto f^{\diamond}$$

that captures the power of an arbitrary but finite number of consecutive applications of f.

- Here we want to consider a new operator

$$f^{\infty} := \ldots \star f \star f$$

that we call inverse limit and that captures the power of an infinite number of consecutive applications of f.

Motivation for considering inverse limits

- Many algorithms on infinite objects are naturally described as infinite loops.
- For instance, a continuous (not necessarily uniquely solvable) initial value problem can be solved on some open interval by WKL.
- With any further application of WKL the domain of the solution can be expanded towards the maximal domain of existence.
- Hence, computing a maximal solution of such an initial value problem is possible with the help of WKL^∞ (Smischliaew 2024).
- But what is WKL^∞?

Different types of loops

- The operators can be seen as descriptions of different types of loops:

operator	loop
f^∞	infinite loop
$f^\diamond$	while loop
$\bigsqcup_{n\in\mathbb{N}} f^{[n]}$	for loop

- Which classes of problems are preserved under what kind of loops?

class of problems	cone	for	while	inifinite	parallel
computable	id	+	+	+	+
finite mind-change computable	$C_{\mathbb{N}}$	+	+	−	−
non-deterministically computable	$C_{2^{\mathbb{N}}}$	+	+	+	+
limit computable	lim	−	−	−	+
Borel computable	$C_{\mathbb{N}^{\mathbb{N}}}$	+	+	+	+

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- Which classes of problems are preserved under what kind of loops?

class of problems	**cone**	**for**	**while**	**inifinite**	**parallel**
computable	id	$+$	$+$	$+$	$+$
finite mind-change computable	$\mathrm{C}_{\mathbb{N}}$	$+$	$+$	$-$	$-$
non-deterministically computable	$\mathrm{C}_{2^{\mathbb{N}}}$	$+$	$+$	$+$	$+$
limit computable	lim	$-$	$-$	$-$	$+$
Borel computable	$\mathrm{C}_{\mathbb{N}^{\mathbb{N}}}$	$+$	$+$	$+$	$+$

Precise definitions

- For simplicity we only consider problems of type $f :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$.
- More general problems can be reduced to this case.
- Using a standard universal function $\mathsf{U} :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ we can define the compositional product

$$f \star g := \langle \mathrm{id} \times f \rangle \circ \mathsf{U} \circ \langle \mathrm{id} \times g \rangle.$$

- We define the inverse limit $f^{\infty} :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ of f by

$$f^{\infty}(q_0) := \{\langle q_0, q_1, q_2, ...\rangle \in \mathbb{N}^{\mathbb{N}} : (\forall i)\ q_{i+1} \in \mathsf{U} \circ \langle \mathrm{id} \times f \rangle(q_i)\}$$

  where $\mathrm{dom}(f^{\infty})$ consists of all $q_0 \in \mathbb{N}^{\mathbb{N}}$ such that
  - $A_0 := \{q_0\} \subseteq \mathrm{dom}(\mathsf{U} \circ \langle \mathrm{id} \times f \rangle)$ and
  - $A_{i+1} := \mathsf{U} \circ \langle \mathrm{id} \times f \rangle(A_i) \subseteq \mathrm{dom}(\mathsf{U} \circ \langle \mathrm{id} \times f \rangle)$ for all $i \in \mathbb{N}$.

Inverse limits as operators

Proposition

$f \leq_{\mathrm{W}} g \Longrightarrow f^\infty \leq_{\mathrm{sW}} g^\infty$.

The problem in proving this is that each loop has to compute the program for the next loop. Technically, this can be handled by an injective recursion theorem.

Theorem (Injective recursion theorem)

Let $f :\subseteq \mathcal{C}(\subseteq \mathbb{N}^\mathbb{N}, \mathbb{N}^\mathbb{N}) \times \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ *be a computable function. Then there is a total computable injection* $R : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ *such that*

$$\mathrm{U}_{R(q)}(p) = f(R, \langle q, p\rangle)$$

for all $q, p \in \mathbb{N}^\mathbb{N}$ *such that* $f(R, \langle q, p\rangle)$ *is defined.*

Examples of inverse limits

- $\mathsf{LPO}^\infty \equiv_{\mathrm{sW}} \mathsf{C}_\mathbb{N}^\infty \equiv_{\mathrm{sW}} \lim$,
- $\mathsf{LLPO}^\infty \equiv_{\mathrm{sW}} \mathsf{K}_\mathbb{N}^\infty \equiv_{\mathrm{sW}} \mathsf{WKL}^\infty \equiv_{\mathrm{sW}} \mathsf{WKL}$,
- $\lim^\infty \equiv_{\mathrm{sW}} \mathsf{J}^{(\omega)}$, where
$$\mathsf{J}^{(\omega)}(p) = \langle p, p', p'', ...\rangle.$$

More generally we have

Proposition (Inverse limits and single-valuedness)

For $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ we define $F^\omega :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ by

$$F^\omega(p) = \langle p, \langle \mathrm{id} \times F\rangle(p), F \star F(p), F \star F \star F(p), ...\rangle$$

for all $p \in \mathbb{N}^\mathbb{N}$ and we obtain $F^\infty \equiv_{\mathrm{sW}} F^\omega$.

Corollary

$f \mapsto f^\infty$ is not a closure operator and neither identical to $f \mapsto f^\diamond$ nor to $f \mapsto \widehat{f}$, nor to $f \mapsto f^\dagger$.

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Countable independent choice

Theorem (B., de Brecht, Pauly 2012)

Let $A, B \subseteq \mathbb{N}^{\mathbb{N}}$ *and let* f, g *be problems. Then*

$$f \leq_{\mathrm{W}} \mathsf{C}_A \text{ and } g \leq_{\mathrm{W}} \mathsf{C}_B \Longrightarrow f \star g \leq_{\mathrm{W}} \mathsf{C}_{A \times B}.$$

Countable independent choice

Theorem

Let $A \subseteq \mathbb{N}^{\mathbb{N}}$ and let f be a problem.

$$f \leq_{\mathrm{W}} \mathsf{C}_A \Longrightarrow f^\infty \leq_{\mathrm{W}} \mathsf{C}_{A^{\mathbb{N}}}.$$

This can be proved as the independent choice theorem for finite compositions (B., de Brecht, Pauly 2012).

Corollary

For every $A \subseteq \mathbb{N}^{\mathbb{N}}$ we obtain:

1. $\mathsf{C}_A^\infty \leq_{\mathrm{W}} \mathsf{C}_{A^{\mathbb{N}}}$ *and* $\mathsf{C}_{A^{\mathbb{N}}}^\infty \equiv_{\mathrm{W}} \mathsf{C}_{A^{\mathbb{N}}}$,
2. $\mathsf{UC}_A^\infty \leq_{\mathrm{W}} \mathsf{UC}_{A^{\mathbb{N}}}$ *and* $\mathsf{UC}_{A^{\mathbb{N}}}^\infty \equiv_{\mathrm{W}} \mathsf{UC}_{A^{\mathbb{N}}}$,

Corollary

$\mathsf{C}_{2^{\mathbb{N}}}^\infty \equiv_{\mathrm{sW}} \mathsf{C}_{2^{\mathbb{N}}}$, $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}^\infty \equiv_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$, $\mathsf{UC}_{\mathbb{N}^{\mathbb{N}}}^\infty \equiv_{\mathrm{sW}} \mathsf{UC}_{\mathbb{N}^{\mathbb{N}}}$ *and* $\mathsf{WKL}^\infty \equiv_{\mathrm{sW}} \mathsf{WKL}$.

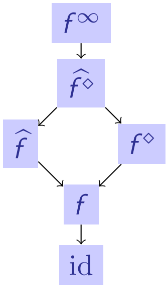
Inverse limits and parallelized diamonds

Question

For which problems does $\widehat{f^\diamond} \equiv_{\mathrm{W}} f^\infty$ hold?

Proposition (Parallelization, inverse limits and diamonds)

1. $\widehat{f^\infty} \equiv_{\mathrm{sW}} f^\infty$,
2. $f^\diamond \leq_{\mathrm{sW}} f^\infty$ *if* $\mathrm{id} \leq_{\mathrm{W}} f$,
3. $\widehat{f^\diamond} \leq_{\mathrm{sW}} f^\infty$, *if* $\mathrm{id} \leq_{\mathrm{W}} f$.



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3. $\widehat{f^\diamond} \leq_{\mathrm{sW}} f^\infty$, *if* $\mathrm{id} \leq_{\mathrm{W}} f$.

Corollary (Inverse limits and single-valuedness)

$F^\infty \equiv_{\mathrm{sW}} F^\omega \equiv_{\mathrm{sW}} \widehat{F^\diamond}$ *for single-valued* $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ *with* $\mathrm{id} \leq_{\mathrm{W}} F$.

Inverse limits of non-computability

Conjecture

$\widehat{\mathrm{NON}^\diamond} <_{\mathrm{W}} \mathrm{NON}^\infty$.

- $\mathrm{NON}(p) := \{q \in \mathbb{N}^\mathbb{N} : q \not\leq_{\mathrm{T}} p\}$.
- NON^∞ can be used to compute countable increasing chains of Turing degrees (above the input).
- $\mathrm{NON}^\diamond$ can be used to compute one finite increasing chain of Turing degrees above the input.
- $\widehat{\mathrm{NON}^\diamond}$ can be used to compute a sequence of finite chains of Turing degrees of arbitrary length above the input.
- The latter can be deduced to the problem of computing a countable infinite decreasing chain of Turing degrees above the input.
- The intuition is that NON^∞ should not be reducible to the latter.

References

- V. Brattka, The Discontinuity Problem, *Journal of Symbolic Logic* 88:3 (2023) 1191–1212
- V. Brattka, M. de Brecht, A. Pauly, Closed Choice and a Uniform Low Basis Theorem, *Annals of Pure and Applied Logic* 163 (2012) 986–1008
- E. Neumann and A. Pauly, A topological view on algebraic computation models, *Journal of Complexity* 44 (2018) 1–22
- H. Smischliaew, Berechenbarkeitseigenschaften von Differentialgleichungen, Fakultät für Informatik, UniBw München 2024
- L. Westrick, A note on the diamond operator, *Computability* 10:2 (2021) 107–110